# How To Get Even

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There are 27 counters in the centre pile. Players take turns claiming 1, 2, 3 or 4 counters from the centre pile. The player with an even number of counters when the centre pile is empty is declared the winner.

When we say we have *solved* a game, we mean we have found a **winning strategy**. What we mean by this, is we have found a strategy that wins every time, assuming both players play optimally. We introduce definitions for **winning positions** and **losing positions** in the following way.

- The empty game (or the game where you have lost and there are no more moves to be made) is a **losing position**.
- A **winning position** is one where **at least one** move sends to a losing position.
- A **losing position** is one where **every** move sends to a winning position.

Every position of the game is either a winning one or a losing one. If you can identify this, you will have the winning strategy from any position of the game. More discussion on winning strategies can be found in Berlekamp, Conway and Guy's wonderful series of books [1]. Let's try working backwards using this idea.

**Solution:**
The first key is to write out every position the game can be in. This includes the parity of each players hand. Here "even even" means both players have already taken an even number of counters, and "even odd" means the first player (one going next) has taken an even number of counters, and the other player has taken an odd number of counters. We write all viable positions as follows, up to 16 counters in the middle:

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| even even | | 1 | | 3 | | 5 | | 7 | | 9 | | 11 | | 13 | | 15 | |
| even odd | 0 | | 2 | | 4 | | 6 | | 8 | | 10 | | 12 | | 14 | | 16 |
| odd even | 0 | | 2 | | 4 | | 6 | | 8 | | 10 | | 12 | | 14 | | 16 |
| odd odd | | 1 | | 3 | | 5 | | 7 | | 9 | | 11 | | 13 | | 15 | |

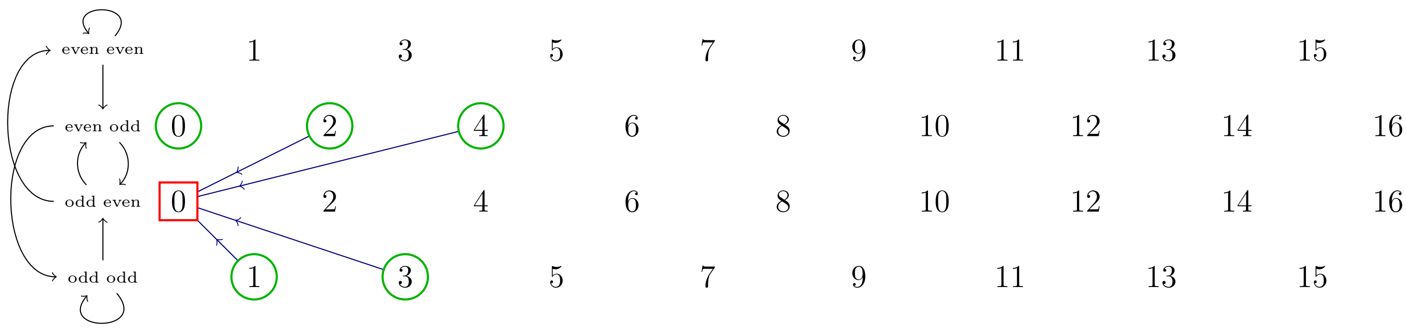
It is easy to see that if there are 0 left and you have even, then you have won, so this is a winning position. If there are 0 left and you have odd, this is a losing position. We put those down:

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| even even | | 1 | | 3 | | 5 | | 7 | | 9 | | 11 | | 13 | | 15 | |
| even odd | 0 | | 2 | | 4 | | 6 | | 8 | | 10 | | 12 | | 14 | | 16 |
| odd even | $\boxed{0}$ | | 2 | | 4 | | 6 | | 8 | | 10 | | 12 | | 14 | | 16 |
| odd odd | | 1 | | 3 | | 5 | | 7 | | 9 | | 11 | | 13 | | 15 | |

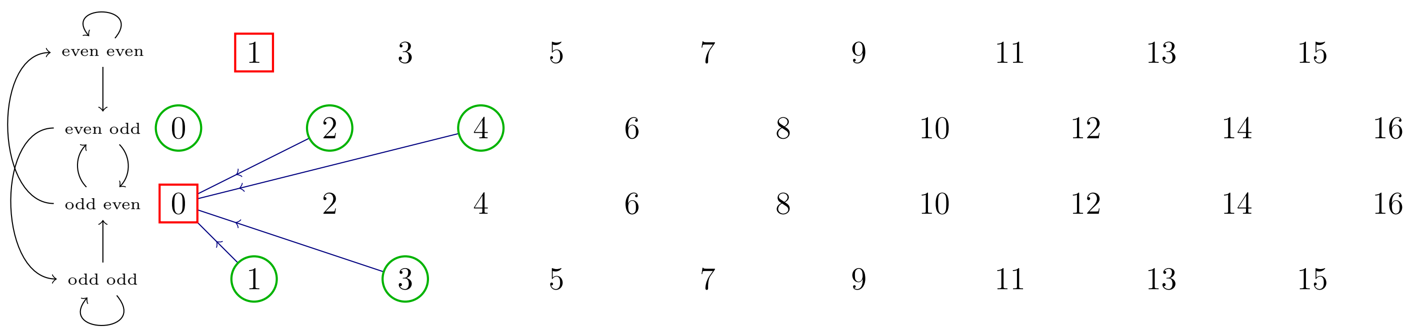
One important detail to be careful of: the board is from the perspective of the player going next, that means after each move, the "even odd" will swap. This means,

| Counters taken | Even Number | Odd Number |
|---|---|---|
| In the First Row | Stay in the First Row | Go to the Second Row |
| In the Second Row | Go to the Third Row | Go to the Fourth Row |
| In the Third Row | Go the the Second Row | Go to the First Row |
| In the Fourth Row | Stay in the Fourth Row | Go to the Third Row |

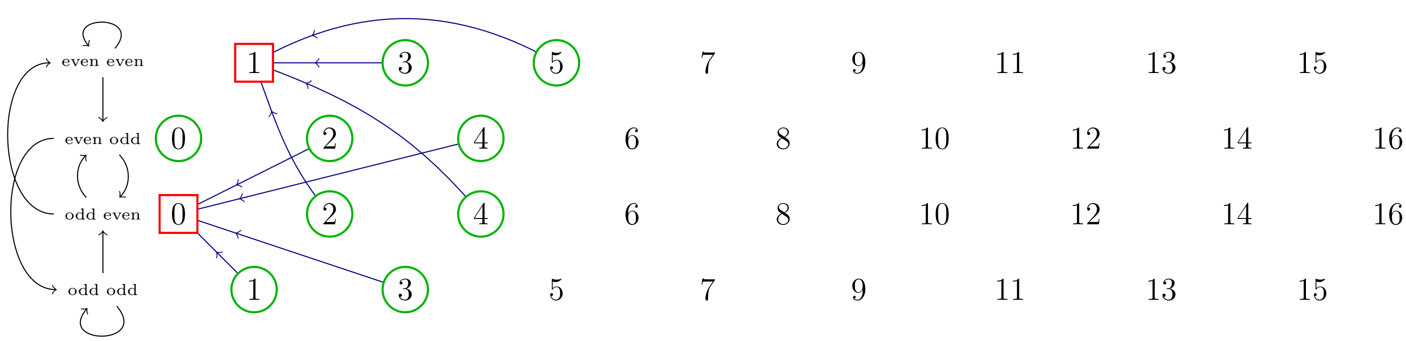
We record this swapping on the left of the diagram. We can also now fill out every position which may send to our first losing position:



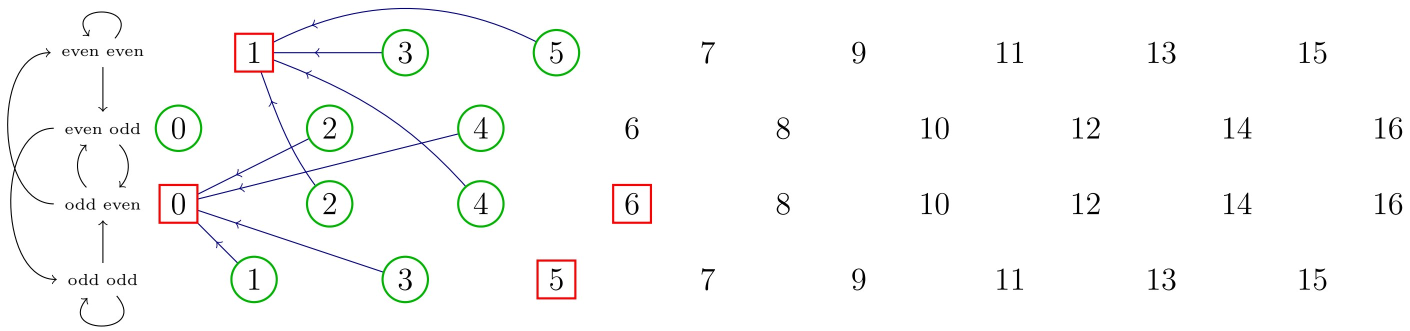

It is also not hard to see that the top left position is surely a losing one, as all it can do is send to a winning position, so we record that data as well:



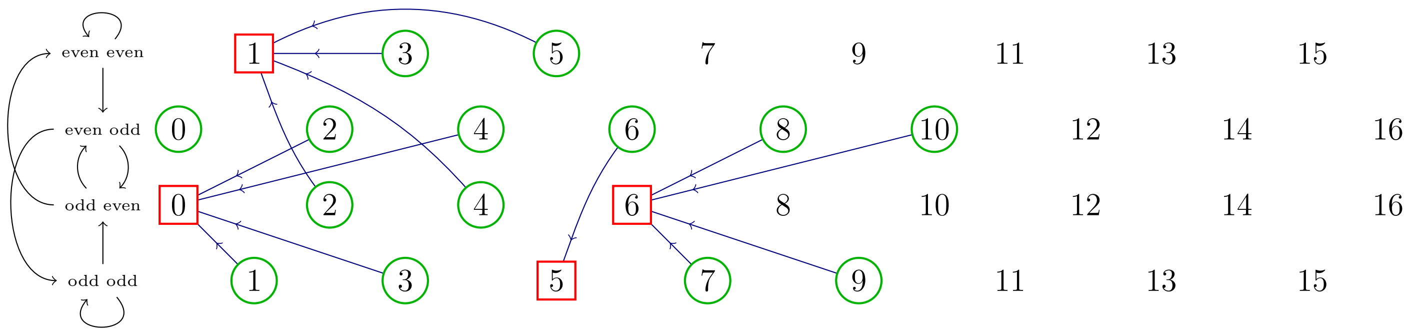

Now we search for all positions which can send to this new losing position, and mark them off as winning positions:



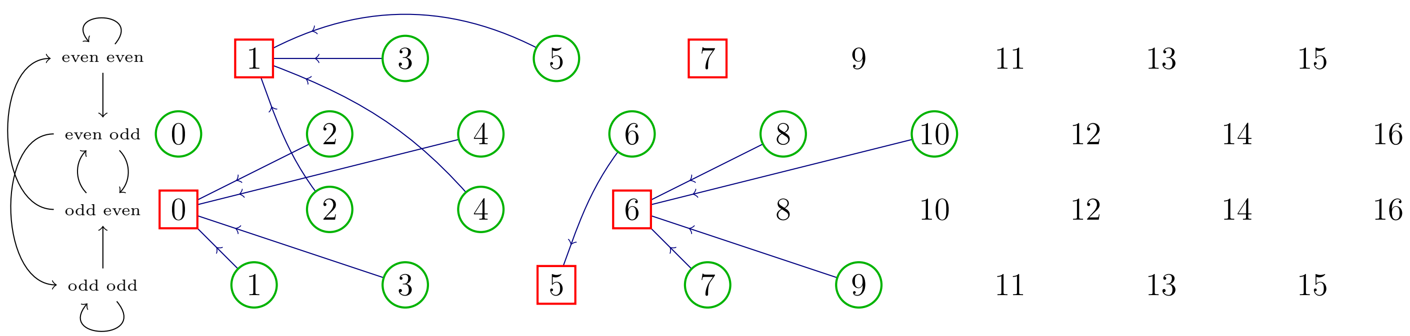

Now we search for a position which can only send to a winning one. It is not hard to see that “odd odd 5” and “odd even 6” are losing positions. We mark them off:



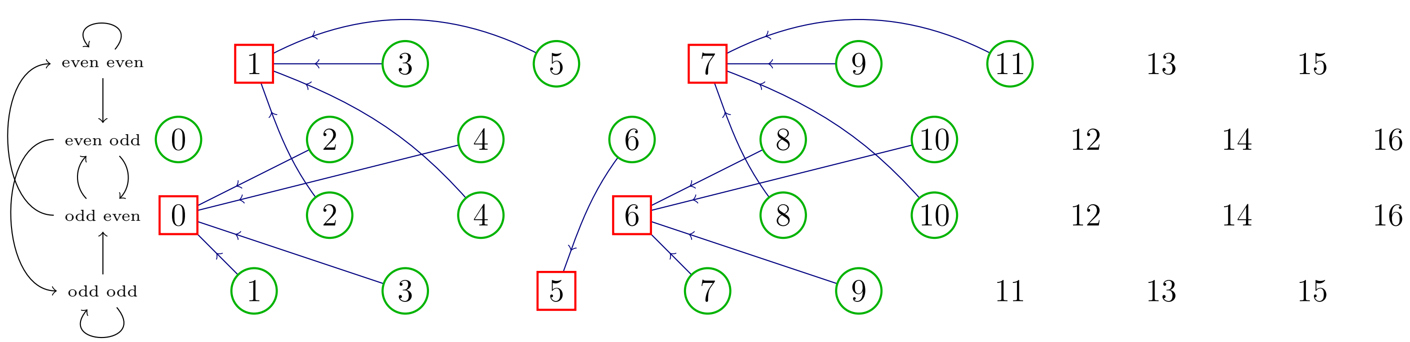
Now we look for every position which which can send to one of our losing positions. We find that "odd odd 7,9" and "even odd 6,8" can send to "odd odd 5", and "even odd 10" can send to "odd even 6". We record these as losing positions:



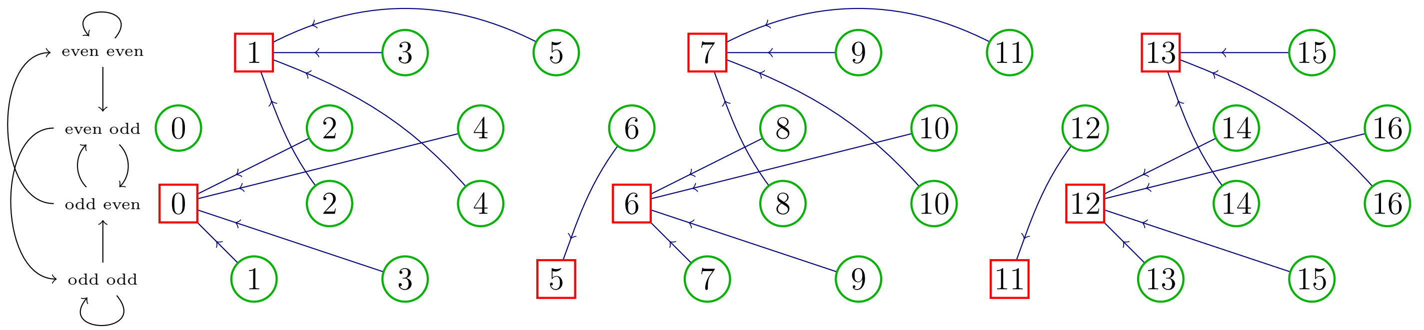
Now we see that "even even 7" can only send to winning positions, so it must be a losing one:



We record every position which can send to "even even 7" as a winning position:

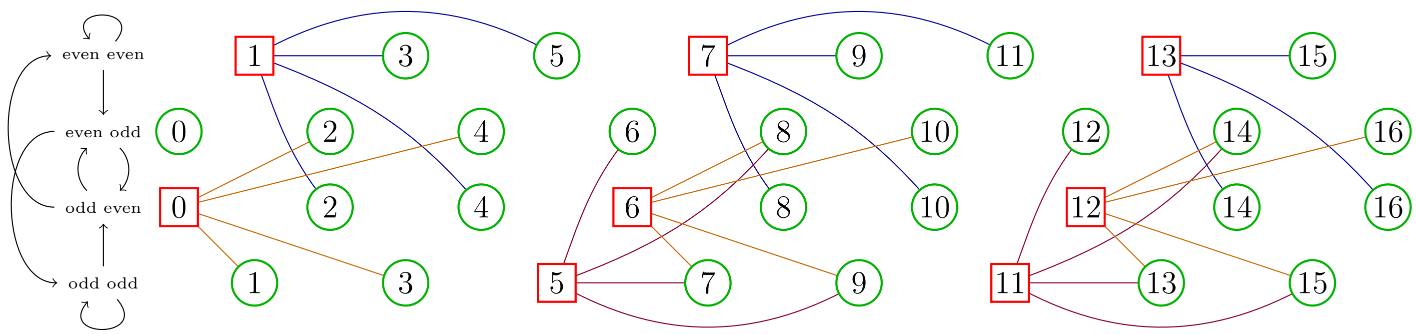


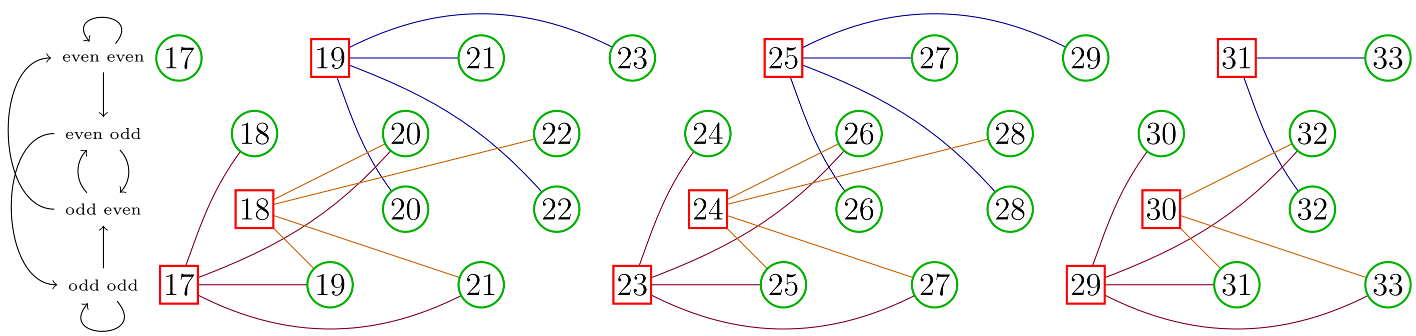
By continuing this process we can fill out the rest of the chart:




Do you see the pattern? The losing positions are the ones which are congruent to 1 (mod 6) where both players have even, congruent to $-1$ (mod 6) where both players have odd, and congruent to 0 (mod 6) where the first player has odd and second player has even.

You may have noticed there are other valid moves that send to losing positions. Here are all the winning moves up to 33 counters. We can easily see that the original game is a 1st player win (even even 27), and the only optimal move is to take 2. We use slightly different colours to indicate if the move is sending to 1 (mod 6), 0 (mod 6), or $-1$ (mod 6).




(note: this diagram is missing the line from even even 17 to even even 13).

We summarize (perhaps more succinctly) the winning strategy below, assuming you are currently in a winning position.

- If you both have even $\Longrightarrow$ send to 1 (mod 6).
- If you have odd, they even $\Longrightarrow$ send to 1 (mod 6).
- Otherwise, $\Longrightarrow$ send to 0 (mod 6) or $-1$ (mod 6).

This is Problem 286 in [2]. You can also consider the misère version of the game where the one ending with an odd number of counters is the winner. Perhaps more interesting is to consider what happens when we vary the number of counters being taken (for example, 1, 2, 3, 4 or 5).

# References


[1] Berlekamp and Conway and Guy, *Winning Ways for your Mathematical Plays*, A K Peters, 2001.

[2] Boris Kordemsky, *The Moscow Puzzles: 359 Mathematical Recreations*, Dover, 1972.