Ms Goncalves and the Breakout Rooms

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The National Museum of Mathematics (in New York City, at the north end of Madison Square Park in Manhattan) has, for two years and ongoing, been running an email puzzle service called "Mindbenders for the Quarantined." Tom Tsao, one of the fifteen thousand or so subscribers, suggested that I, as puzzle supplier, compose a puzzle based on breakout rooms.

The following proved to be quite a popular entry, perhaps one that Martin Gardner would have liked for his famous *Mathematical Games* column. It goes like this.

"Each day Ms Goncalves distributes her twelve fifth-graders into Zoom breakout rooms containing three or four students each. She has devised a schedule in which every pair of students is together in a breakout room exactly once.

How many days does her schedule run?"

As usual for Mindbenders, the puzzle appeared on a Sunday morning; the following Tuesday there was a hint:

"How many pairs of students are there in all? How many are together on a given day?"

and then on Thursday, a bigger hint:

"Show there are only two numbers of days that make the pairs come out right, and one of them is impossible."

Below, we expand a bit on the offered solution, with the object of illustrating how puzzles like this are most effectively tackled.

For the purposes of this puzzle, a "pair" of students is an *unordered* pair; that is, {Alice, Bob} is the same pair as {Bob, Alice}. It's actually easier

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to count *ordered* pairs, since (in this case) there are 12 choices for the first student in the pair, then 11 for the second; thus, $12 \times 11 = 132$ ordered pairs in all. Since this counts every unordered pair (like {Alice, Bob}) twice, we have to divide by 2 to get the number of unordered pairs: 66.

In general, if there are n objects, then the number of unordered pairs you can form from them is $n(n-1)/2$, often expressed by mathematicians as "n choose 2."

The puzzle statement stipulates that each of the 66 pairs of students appears together exactly once. It's natural to ask: how many pairs are "taken care of" in a day?

There are only two ways to size the breakout rooms on a given day. A "Type A" day, say, has three breakout rooms of size 4 each; a "Type B" day has four breakout rooms of size 3 instead.

On a Type A day, each room has 4 students and thus services 4 choose 2 $= (4 \times 3)/2 = 6$ pairs; thus, the three rooms take care of $3 \times 6 = 18$ pairs of students altogether.

On a Type B day, each room services only $(3 \times 2)/2 = 3$ pairs of students, so the four rooms take care of $4 \times 3 = 12$ students altogether.

It follows that if Ms Goncalves' schedule unites each of the 66 pairs of students exactly once, and if it is comprised of a Type A rooms and b Type B rooms, then we must have:

$$a \times 18 + b \times 12 = 66.$$

To figure out what pairs of values are possible for the numbers a and b, it's useful to divide that equation by 6 to get:

$$a \times 3 + b \times 2 = 11.$$

Now it's pretty each to check that the only possibilities are $a = 3$ and $b = 1$, or $a = 1$ and $b = 4$. Since the schedule runs for $a + b$ days, we see that in the first case, it's a 4-day schedule; in the second, a 5-day schedule. Well, which is it, then?

Suppose it's a 4-day schedule, which has three days with breakout rooms of size 4. Say one of those days is Monday, and another Wednesday. Consider one of Wednesday's breakout rooms; say it contains Carla, George, Miguel and Sumit. Since there were only three breakout rooms on Monday, two (at least) of these four students must have been in the same room on Monday. (Technically speaking, this is an application of the *Pigeonhole Principle*, which says that if n pigeons occupy fewer than n holes, then some hole must contain at least two pigeons.) But then that pair was together on both Monday and Wednesday, which is not allowed.

We conclude that no schedule can contain more than one Type A day, and in particular, a four-day schedule (which contains three Type A days) is not possible. Hence, the answer to the puzzle is that Ms Golzalvez' schedule runs for five days.

Wait—if you don't trust the puzzle-poser (and you certainly should not, in this case), you'll want to confirm that there really is a five-day schedule that works. Constructing one is a "hammer and tongs" process, at least for me, but it's not a difficult one. Labelling the students by letters A through L, you can certainly assume without any loss of generality that on the unique Type A day in the schedule, the rooms are $\{A, B, C, D\}$, $\{E, F, G, H\}$, and $\{I, J, K, L\}$. The next day's rooms will need to take one student from each of these groups, so the rooms may as well be $\{A, E, I\}$, $\{B, F, J\}$, $\{C, G, K\}$, and $\{D, H, L\}$. After this some blind alleys are possible, but after a few tries you'll end up with something like this:

Day 1:	ABCD	EFGH	IJKL	
Day 2:	AEI	BFJ	CGK	DHL
Day 3:	AFK	BEL	CHI	DGJ
Day 4:	AGL	BHK	CEJ	DFI
Day 5:	AHJ	BGI	CFL	DEK

Your table may differ—there are many solutions. Next step: let's put an end to the COVID pandemic and break out of breakout rooms!