Some New Magic With
"Most-Perfect" Magic Squares

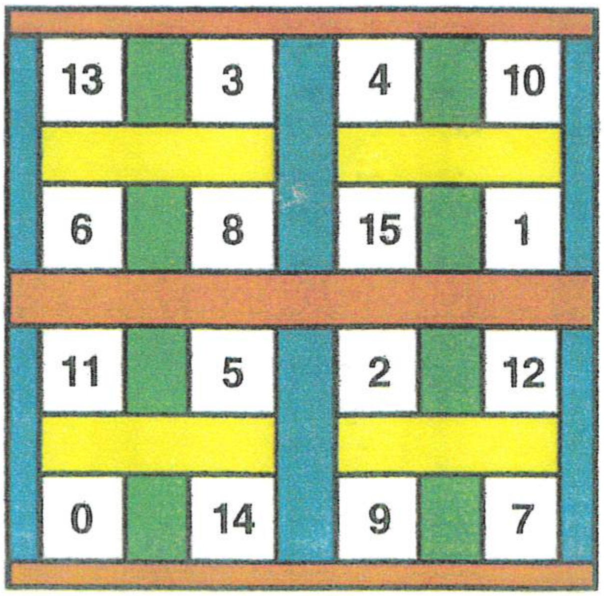
By Jeremiah Farrell

In 1998 in his article "Magic Squares Cornered" appearing in NATURE (1) Martin Gardner reports

> Dame Kathleen Ollerenshaw, one of England's national treasures, has solved a long-standing, extremely difficult problem involving the construction and enumeration of a certain type of magic square. The solution comes in a book written with David Brée. (2)

Dame Kathleen Ollerenshaw (1912-2014)
D.B.E., D.S.U., D.L., C. Math.

Most-perfect squares of order n x n have three properties. One, they are pandiagonal which means every row, column and ALL diagonals, including the broken ones sum to the same constant. Secondly, every 2x2 sub-square must sum to this constant and thirdly every pair on a diagonal ½n apart sum to half the constant. These properties force the order n to be 4m, m=1, 2,. . . . Hence the first example is of order 4x4 and the second of 8x8. Using the 16 numbers 0, 1,…,15 for n=4 and the 64 numbers 0,1,…,63 for n=8 yields these two examples among others. The constant in either is $2(n^2-1)$ and the diagonal hop sums to n^2-1.

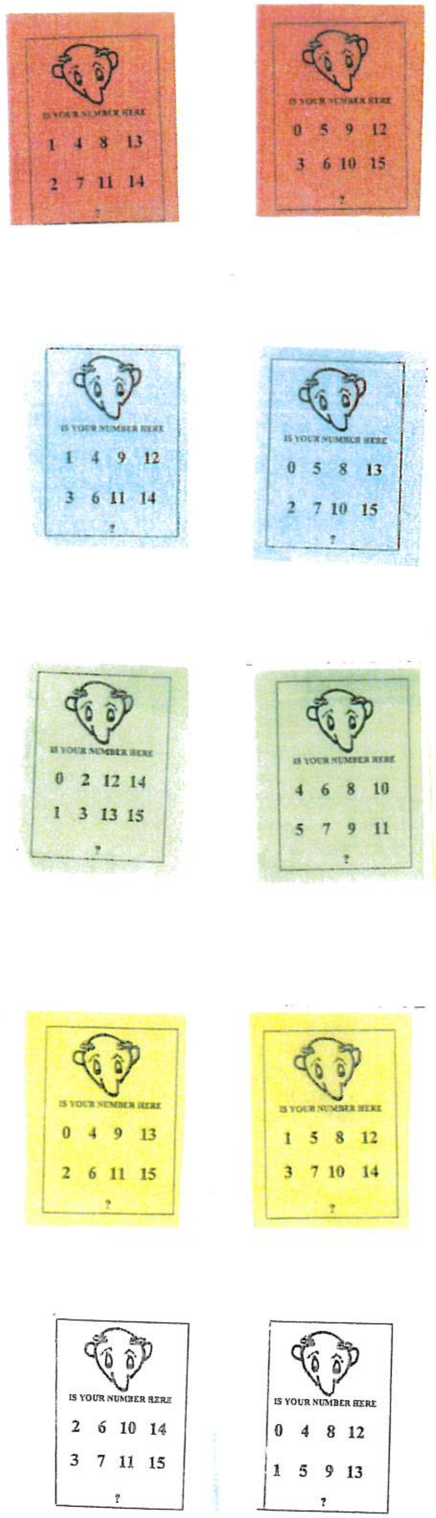


Constant = 30

Diagonal hop sums to 15

Constant = 126

Diagonal hop sums to 63



Our new magic performed on the 4x4 square starts with noting that the square can be regarded as a torus by bending the top red around to join the bottom red and joining the left and right blues to complete the doughnut shape. We supply five colored cards with all the numbers shown on either front or back. Now no matter how the cards are actually turned there will always be exactly one number showing on all five cards or not showing on all five cards. This number is the "Key" and can be changed to another by flipping the cards. As the cards lie here the Key=5.

The subject is to privately choose one of the 15 numbers and secretly choose to tell the truth to all questions or to lie to all questions. Once the subject's answers are given the magician looks quickly at the most-perfect square and correctly names the selection.

METHOD: The magician knows the Key, here 5, and traces the Yes (or No) response from the Key. The white card denotes a diagonal hop. For example suppose the left sides are showing and 4 is chosen. Telling the truth yields red=yes, blue=yes, and yellow=yes. The magician starts at the Key=5, jumps, say, yellow to 14 then red to 3 and blue to 4. Note that if the subject lies the yeses would be green to 11 and white a hop to 4.

The next page depicts a common two dimensional drawing of a four dimensional hypercube along with the "most-perfect" magic square. The parts of the 4-cube are given by the generating function $y = (1+2x)^n$. For n=4 this gives

$$y=1+8x+24x^2+32x^3+16x^4$$

Thus there are in the 4-cube 16 points, 32 lines, 24 squares, 8 cubes and 1 hypercube. All these parts may be found exactly in the magic square as follows. Turn the square into a torus and each of the 16 numbers forms a 2x2 square with each number at the bottom right. For example

7-0 or 0-14
10-13 13-3

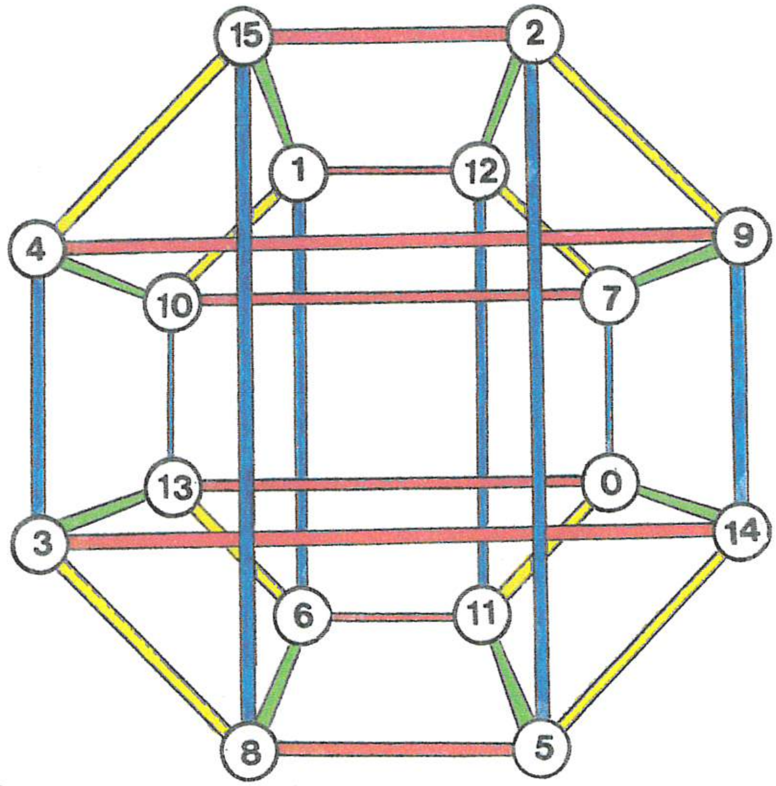
Down to 2 – 12
9 – 7

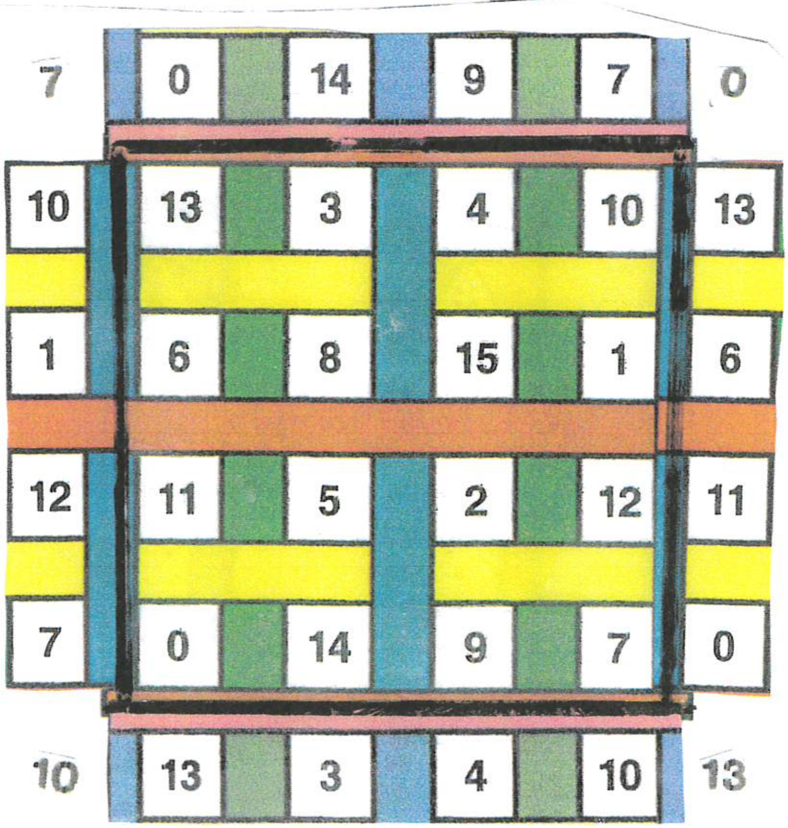
Also each of the four rows of 4 and columns of 4 yield 8 more 2x2 squares. The 8 cubes are formed from the magic square using the four double rows and the four double columns. For example

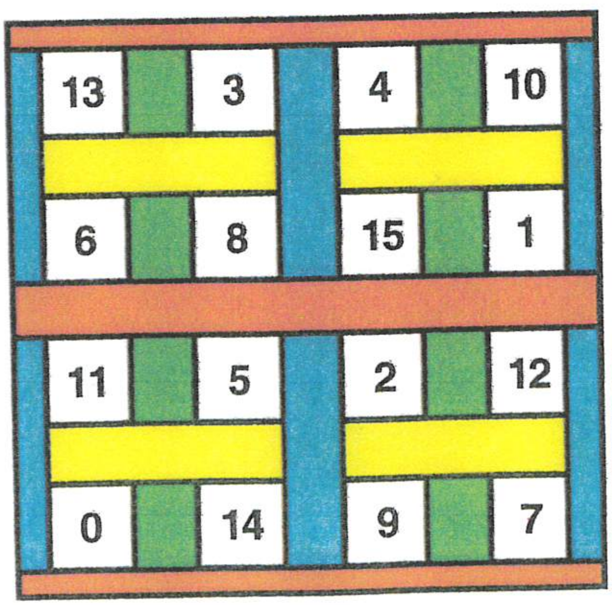
0-14-9-7 The 32 colored lines are obvious.
13-3-4-10

It may be possible to obtain other larger hypercubes from higher order most-perfect squares.

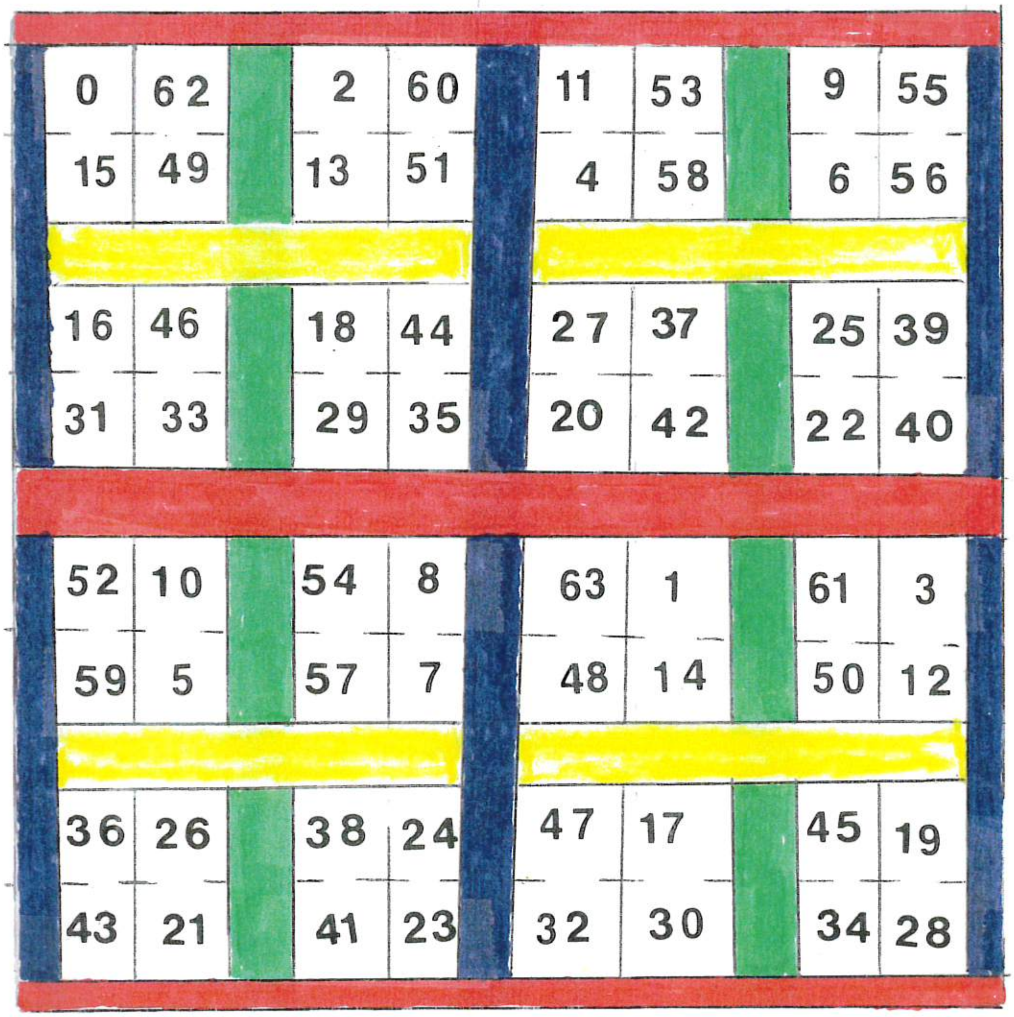
There is also an extension of the 4-cube to 5 dimensions. Notice on the magic square the unique diagonal hops of pairs that sum to 15: 0-15, 1-14, 2-13,...7-8. This turns the square into half a 5-cube. The magic can now be performed using the five cards shown. As usual no matter how the cards are turned there is always exactly one number, the key, that appears on all five. The cards displayed have key=5. When the subject secretly chooses a number and privately chooses to either lie to all cards or tell the truth to all, the magician can quickly locate the subject's selection from the magic square. For instance, suppose 8 was selected and the subject chooses to lie. In any order her responses would be with left sides up: Red No, Blue Yes, Green Yes, Yellow Yes, and White Yes. From the key=5 we trace 5 blue to 2 green to 12 yellow to 7 hop to 8.


15
2
1
12
4
9
10
7
13
0
3
14
6
11
8
5


7 0 14 9 7 0
10 13 3 4 10 13
1 6 8 15 1 6
12 11 5 2 12 11
7 0 14 9 7 0
10 13 3 4 10 13


13 3 4 10
6 8 15 1
11 5 2 12
0 14 9 7

To perform similar magic on an 8x8 most-perfect square we start by using the following square and the five colored cards on red, blue, green, yellow and white.



These cards each have half of the 64 numbers 0, 1, 2, . . . 63 on in special ways and towards the end of our trick we will ask the questions to be answered truthfully by the subject. (1) Is your number even? (2) Is your number 32 or greater?

The colored cards are

0	18	34	48
1	19	35	49
4	22	38	52
5	23	39	53
10	24	40	58
11	25	41	59
14	28	44	62
15	29	45	63

2	16	32	50
3	17	33	51
4	22	38	52
5	23	39	53
10	24	40	58
11	25	41	59
12	30	46	60
13	31	47	61

0	20	36	48
1	21	37	49
2	22	38	50
3	23	39	51
12	24	40	60
13	25	41	61
14	26	42	62
15	27	43	63

0	16	32	48
1	17	33	49
4	20	36	52
5	21	37	53
10	26	42	58
11	27	43	59
14	30	46	62
15	31	47	63

1	16	33	48
2	19	34	51
5	20	37	52
6	23	38	55
9	24	41	56
10	27	42	59
13	28	45	60
14	31	46	63

The subject is to secretly choose a number from the 64 numbers and separate the five colored cards into two piles, one with the choice on each and the other without the choice on them. Then the subject tells correctly the answer to the two questions. By looking at the most-perfect square the magician correctly names the choice.

METHOD. The 8x8 is turned into a torus similar to the 4x4 case. Either pile of colored cards will locate the correct 2x2 square containing the subject's choice on starting at the Key 54-8.
57-7

The white card is a single diagram hop. When the answers to the two questions are known the choice is identified.

As an example suppose 37 is the choice. One pile of colored cards contains the green, yellow and white cards and the other pile red and blue. Using either pile the magician finds the current 2x2. From the Key 54-8 trace green to 52-10,
57-7 59-5
yellow to 36-26 and then white
43-21
A hop to 27-37, and locates
20-42
the choice 37 once the two questions are answered.

The reader will note that this magic is similar but much harder to fathom than the old chestnut trick using base two cards.

Martin Gardner remarks that the authors for the first time were able to find all the most-perfect squares of all orders. For example, not counting reflections or rotations there are 48 4x4s and 368640 8x8s. When you reach 36x36 the number is 2.76754 x 10^{44} - around a thousand times the number of pico-pico-seconds since the Big Bang. Gardner adds

> This solution of one of the most frustrating problems in magic-square theory is an achievement that would have been remarkable for a mathematician of any age. In Dame Kathleen's case it is even more remarkable, because she was 85 when she and Brée finally proved the conjectures she had earlier made. In her own words, "The manner in which each successive application of the properties of the binomial coefficients that characterize the Pascal triangle led to the solution will always remain one of the most magical mathematical revelations that I have been fortunate enough to experience. That this should have been afforded to someone who had, with a few exceptions, been out of active mathematics research for over 40 years will, I hope, encourage others. The delight of discovery is not a privilege reserved soley for the young."

Perhaps the reader would prefer an alternative to the two questions that must be answered truthfully in the 8x8 case. This can be accomplished by using the following two orange cards instead.

0	16	36	52
1	17	37	53
2	18	38	54
3	19	39	55
8	24	44	60
9	25	45	61
10	26	46	62
11	27	47	63

0	16	32	48
2	18	34	50
4	20	36	52
6	22	38	54
9	25	41	57
11	27	43	59
13	29	45	61
15	31	47	63

They are to be added to the two piles of five cards under the same provisos. The key 2x2 with 7 as the lower right is transposed by either pile into another pile (unless one of the four members of the key is chosen) and the magician traces the new 2x2 from the lower right across the oranges with the solid or dashed sides. That is, the four entries of the 2x2 will be, starting at the lower right entry as follows. Neither orange stays on lower right, both oranges cross the dashed and solid lines, and one of the oranges goes across only the dashed or solid lines. If that new entry occurs on any of the colors that is the subject's choice. If not, then this is the "not" pile and the subject's choice is the diagonally opposite choice.

For example suppose the subject chooses 39. The two piles will then be first red blue, green, dashed and second yellow, white, solid. If the magician traces the first pile it goes from key 7 to 39 and from the second pile it would lead to 22 but 22 does not appear on either yellow, white or solid so this is the "not" pile and the subject's choice is the diagonal 39 instead.

In summary, the red, blue, green, yellow and white cards identify the proper 2x2. Then the oranges locate one of the four entries starting at the lower right. If the pile is noted to be the "hit" pile that entry is the choice. If the pile is noted to be the "not" pile, the diagonal is the choice.

REFERENCES

(1) Magic Squares Cornered", Martin Gardner, NATURE, vol. 395, 17 September 1998.

(2) *Most Perfect Pandigonal Magic Squares, Their Construction and Enumeration*, by Kathleen Ollerenshaw and David Brée. ISBN 0 90501 06X. From The Institute of Mathematics and Its Applications, Catherine Richards house, 16 Nelson Street, Southend-on-Sea, Essex, SS1 1EF U.K.

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Martin Gardner and I in Norman, OK, 2004