

Tetradecahedron as Palimpsest of the Monododecahedral 1-Parameter Family of the Polymorphic Elastegrity

Subtitle: Making Paper Bubbles

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Introduction: Origins of the structure from two Bauhaus basic design exercises

In this article, we review and expand upon earlier G4G publications^{1,2} of the polymorphic elastegrity structure that was discovered through paper folding and weaving. Two basic design exercises at the Yale Architecture School led to the discovery of a “paper diamond”.

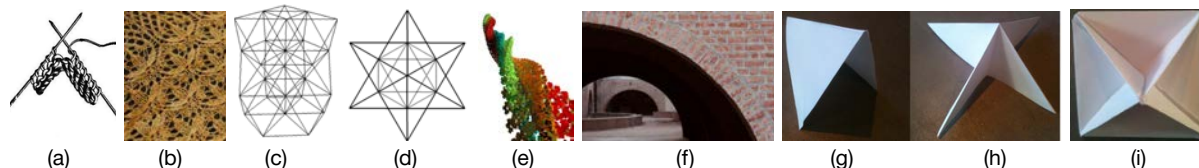


Fig. 1: 1971 exercise (a) Students were told that design starts by finding simple rules. Anni Albers of the Bauhaus likened it to knitting where the simple rules are to place the yarn over or under a needle. (b) Interest and complexity result from using simple rules. (c) & (d) The simple rules given to students were the only two ways that exist to close pack spheres of equal diameters, as represented with applicator sticks arranged in octahedral-tetrahedral lattices. The vertices represent the centers of spheres that can only be close-packed as (c) A-B, repeating every second layer, and (d) A-B-C, repeating every third layer. Students were also told that 100% of the periodic table crystals are homologous to one or the other close packings, and were instructed to use these two ways of close packing spheres to create interest. (e) A helix with grooves to grow branch helices as seen in the digital recreation of the 1971 finding, resulted from the mechanical repetition of A-B, A-B-C, A-B, and so on. **1972 exercise** (f) Louis Kahn, the famous architect, said to brick, “What do you want, brick?” Brick says to you, “I like an arch.”³ Students were told to allow material to dictate form, in the spirit of Joseph Alpers material exercises;⁴ (g) A diagonal crease on a square piece of paper became surprisingly stable and raised the question of what would happen if a second diagonal was added; (h) A second crease created a pyramid. It was recognized as an octahedral fragment due to the familiarity with octahedra gained with the 1971 exercise. It made one wonder, could several paper pyramids make a whole octahedron, and how many? (i) Experimenting with paper showed “paper liked an octahedron” as six crosses of triangles could be assembled into a stable octahedron, by placing two triangles of one axis over, and two triangles of the other axis under adjacent crosses. The resulting octahedron was named a “paper diamond” because it was hard, the number six suggested carbon six, and uncut diamonds come as octahedra.

Others independently invented what is named paper diamond here, and called it various names.⁵ Diamonds, though, are crystals and crystals grow. Having named it paper diamond, a quest started how to grow paper crystals. Two ways of paper crystal growing were found, one with face connectors fig 2(a), (b), (c) and another with edge connectors fig. 2(e), (f), (g), (h), (i).

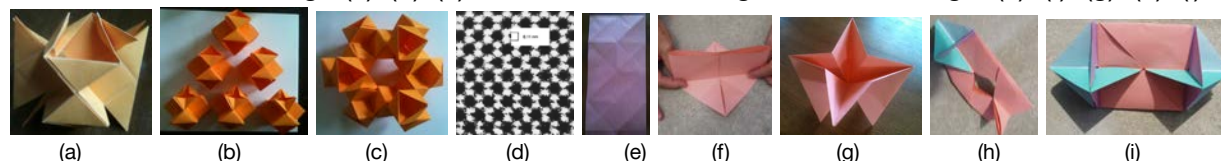


Fig. 2 Face connectors (a) Create a four-cube-corner unit by folding four of the eight octahedral faces; (b) Six four-cube-corner units; (c) Units assembled by inserting a cube into the missing corner creating a strong bond to grow a crystal; (d) Actual diamond under an electron microscope resembles the paper analog of the crystal; **edge connectors** (e) A two-square rectangle with diagonals and crosses creased through the centers of the squares is the element used to create a pyramid with insertable wings; (f) Make a square on the diagonal; (g) Fold the square into a pyramid with wings; (h) Wings inserted in each other to grow the crystal; (i). The pyramids are woven into paper crystals with edge connectors growing the crystal with malleable connections.

A failed experiment leads to the discovery of the Polymorphic Elastegrity

Experimenting to simplify the assembly and make sturdier paper crystals led to further explorations. Creating edge connectors 3(i) as we saw, required a two-square rectangle where each square had a cross creased and two diagonals through their centers. A slit was torn

between the two centers of the squares fig 3(a), attempting to discover an improved way of linking crystal units.

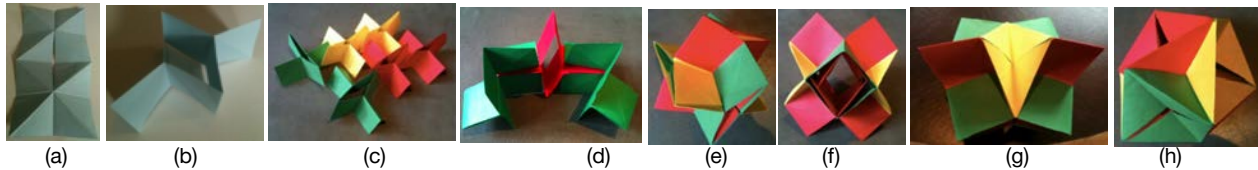


Fig. 3 Discovery of the Polymorphic Elastegrity (a) A slit is torn between the centers of the pre-creased squares; (b) The rectangle is folded axially in half and squeezed to create a cross of eight little squares; (c) One axis of the cross has four squares with a closed ridge on top and open sides. The other axis has four little squares, open on top forming a slit and closed sides; (c) Six units; (d) Weave units placing the side of the cross with the open slit over a side with a ridge on top; (e) Weaving the six crosses of little squares over and under as in step (d), results in three large intersecting squares that do not stay tight together. The slits remain gaping open. It was discarded as a failed experiment. Having forgotten this failed experiment, this flaccid structure was woven again a few months later. (f) Attempting to salvage time spent creating it, the slits were opened; (g) The little squares with the slit on top were folded in half on their creased diagonals into two right triangles hinged along a leg. (h) When all six slits are opened and inverted, twelve elastic hinge systems stabilize the entire structure into an icosahedron. Each hinge system consists of the two right triangles hinged along a leg, with their hypotenuse elastically hinged to a tetrahedron. Each tetrahedron is elastically supported by three hinge systems that link it to three tetrahedra that rotate with opposite chirality.

The resulting structure fig. 3(h) has four pairs of tetrahedra along four axes AA', BB', CC', and DD' levitating on six pairs of elastic hinge systems. Each hinge system consists of two right triangles hinged to each other along a leg (shown in green) and along their hypotenuses to two tetrahedra (shown in red) fig. 4(e). Each pair of hinge systems surrounds a gate with four free legs fig. 4(d) that open and close around three orthogonal axes 1, 2, and 3 fig. 4 (a), (b), (c).

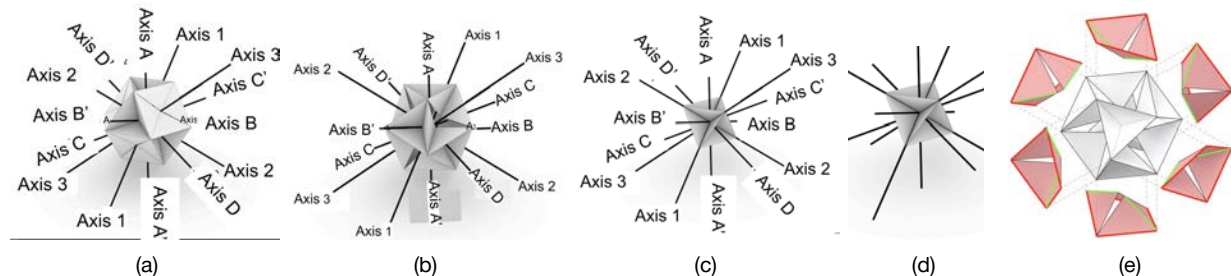


Fig. 4 Polymorphic Elastegrity (a), (b), (c) Four tetrahedral axes and three orthogonal gate axes that do not move. (a) Expanded into a cuboctahedron, leg hinge dihedral angles 180° , hypotenuse hinge 70.52° ; (b) Regular icosahedron, leg hinge dihedral angles 90° , and hypotenuse hinge dihedral angles 28.72° ; (c) Contracted into an octahedron, all thirty-six dihedral angles 0° ; (d) Seven non-moving axes all motion is in relation to them; (e) Six gates that open and close around the three orthogonal axes.

When a force is applied along any of the four tetrahedral axes fig. 5, it actuates all thirty-six hinges simultaneously. Twelve leg hinges (red), and twenty-four hypotenuse hinges (green) fig. 4(e), open and close symmetrically and in sync around the three axes fig. 4(c & d). The gates close as the dihedral angles of their leg hinges expand cooperatively to 180° , the hypotenuse hinge angles expand to 70.52° , and the structure turns into a cuboctahedron with closed gates fig. 6 (a). The six gates open in sync in response to compression along any of the tetrahedral axes. When the leg hinge dihedral angles reach 90° the twelve vertices outline a regular icosahedron fig. 6(c). The gates reach their maximum opening at dihedral angles 77.18° .⁶ The gates close again as dihedral angles contract to 0° and turn the structure into an octahedron fig. 6. The tetrahedra rotate in sync as they slide along the tetrahedral axes towards or away from the center, and gates close and open and close again. Four tetrahedra rotate chirally and four tetrahedra located diametrically opposite rotate anti-chirally.

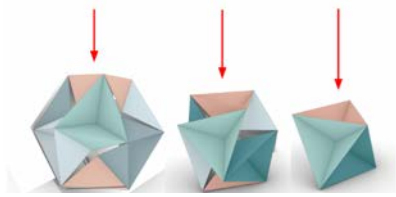


Fig. 5 A force is applied on one of the four tetrahedral axes expand

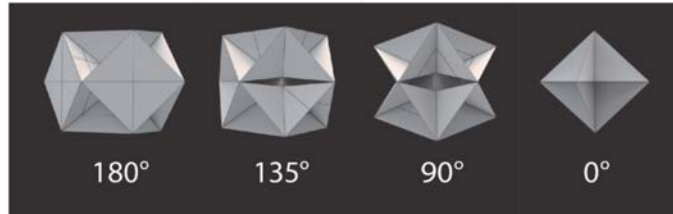


Fig. 6 Gates open and close as the twelve leg hinge dihedral angles to 180° cuboctahedron, 90° icosahedron, and contract to 0° octahedron

The asymmetrical tetrahedra form a resilient structure that keeps its shape in elastic equilibrium fig 3(h). When a force deforming is removed, it springs back to its original shape. This is the reason that it was named elastegrity by analogy to tensegrity, which also maintains the integrity of the shape through pre-tension by springing back when a deforming force is removed.

This structure was previously reported at G4G under different names. In 2016 for G4G12 it yielded a mono-dodecahedron, that is a polyhedron with **twelve** congruent, but not necessarily regular faces. In 2018 appropriately for G4G13, its **thirteen** axes were reported. At that time it was still known as chiral icosahedral hinge elastegrity. An editor renamed it Pavlides Elastegrity in 2020 simplifying the name and arguing that structures invented by architects such as the Hoberman Sphere, and the Rubik’s Cube are named after the architect who invented them. And since the editor, Elidir King, was classically trained, he also pointed out that Archimedes, inventor of the Archimedes screw, was an architect, the naval architect of Syracuseia the largest boat ever constructed in antiquity, as well as an engineer and a mathematician, as Archimedes is more commonly known.



Fig. 7 Architects who invented structures: (a) Renamed in 2020 Pavlides Elastegrity, (b) Hoberman Sphere, (c) Rubik’s cube (d) Archimedes Screw, Architect of the greatest vessel in the antiquity cruise ship, battleship, and freight ship all in one.

However, in 2022, the structure was renamed Polymorphic Elastegrity, due to its shape-shifting properties. First, it contracts into an octahedron and expands into a cuboctahedron, as we saw above fig. 6. With further folding, it can flatten into a multiply covered square and morph into shapes with the vertices of each of the Platonic shapes as presented at G4G12.¹

The monododecahedral path

The polymorphic elastegrity, through further folding, turns into a monododecahedron Fig. 8(f),

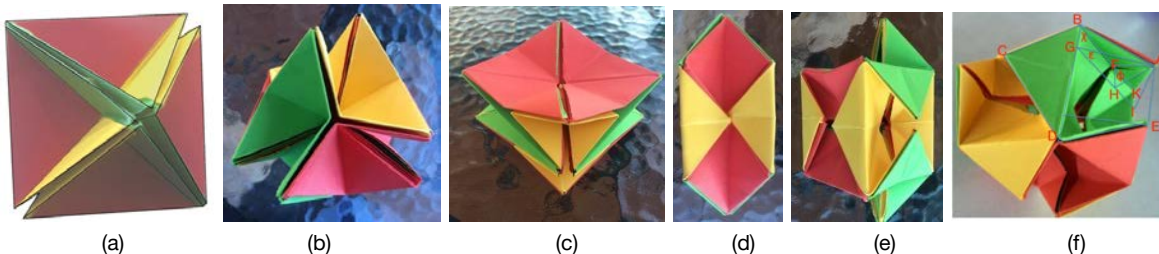


Fig. 8 Shape shifting through further folding. (a) Eight tetrahedra contracted into an octahedron; b) The eight rigid tetrahedra are crushed with their right triangle faces bisected through creasing, creating eight groups of triradiational triangles. (c) The

triradiational triangles open as shown into flat squares subdivided into four little squares through the creases; (d) Fold the diagonals of the little squares into triangular flaps to cover the slits; (e) The twelve folded flaps create eight pinwheels around the collapsed centers of the tetrahedral equilateral faces, raising the structure into a cube; (f) By lifting the flaps from the face of the cube at a dihedral angle ϕ and angle ϵ created by $AG \perp BC$ so that $\phi + \epsilon = 90^\circ \Rightarrow \sin \phi + \cos \epsilon = 1$

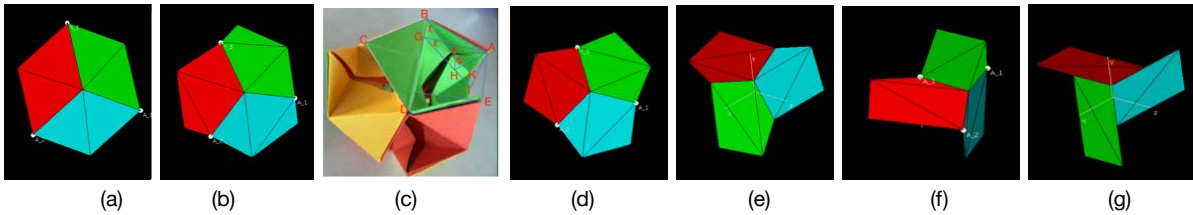
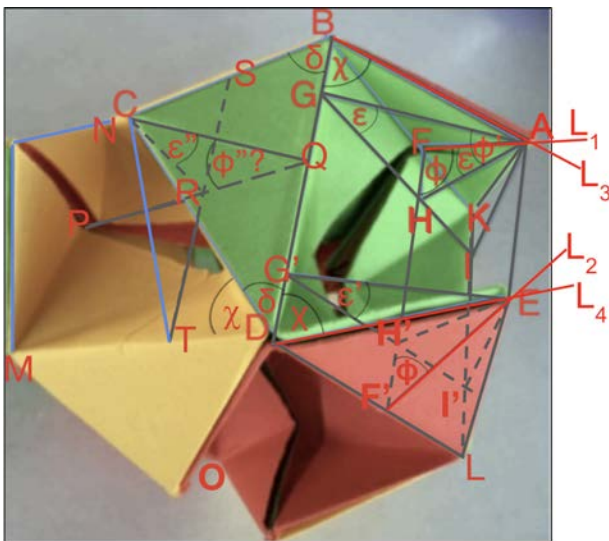


Fig. 9 Monododecahedral path - computer animations by Thomas Banchoff : (a) Rhombic (degenerate pentagons one side is 0); (b) Pentagons one angle is smaller than 90° , (c) Pentagon has one right angle; (d) Regular dodecahedron; (e) Pentagon one angle is greater than 108° ; (f) Pentagons one angle is much greater than 108° ; (g) Rectangle (degenerate pentagon has one angle 180°)

Math proof how to construct a monododecahedron on a cube for any dihedral angle ϕ :



Use fig. 10 to support the proof that for every dihedral $\angle \phi$ there is a $\angle \theta = \angle BAK = \angle DEL = \angle BCD$ so that ABCDE is flat. What we have is the central cube of the elastegrity in fig. 10, (in what we call the mono-dodecahedron position). What we are looking for is a flap's shape and position such that the vertices of the flaps together with the vertices of the cube form a monododecahedron. The cube is given, the angle of the flap is given. What we need to figure out is where to position the vertices of the flaps in such a way that the resulting figure is a mono-dodecahedron. In particular, ABCDE needs to be flat. (In the physical object the correct positions for the vertices can be realized by further folding the flaps). Draw both planes with dihedral $\angle \phi$ to face $\square BDLK$: (a) plane 1a on through BK, that will contain flap $\triangle ABK$ once A is fixed, and (b) plane 1b through DL that will contain flap $\triangle DEL$, once E is fixed; Draw plane 2 \perp $\square BDLK$ bisecting it w/ FF' ;

Fig. 10 Monododecahedron

Intersect plane 1a & plane 1b w/ plane 2 creating respectively lines L_1 where AF will lie once A is fixed and L_2 where EF' will lie once E is fixed; Draw plane 3 through edge BD w/ dihedral to $\square BDLK$ $\angle \epsilon = 90^\circ - \angle \phi$; Intersect plane 3 w/ plane 1a & plane 1b creating lines L_3 where AB will lie and L_4 where DE will lie once A and E are respectively fixed; Intersect L_1 w/ L_3 to fix point A, and intersect L_2 w/ L_4 to fix point E; Draw AH & $EH' \perp \square BDLK$ & $GH \parallel G'H' \parallel BK \parallel DL$; $\triangle AGH \cong \triangle EG'H'$ because (a) $\angle \epsilon = \angle AGH = \angle AG'H'$; (b) $GH = G'H'$; (c) $\angle AHG = \angle EH'G' = 90^\circ$; $\triangle AFH \cong \triangle EF'H'$ because (a) $\angle \phi = \angle AGH = \angle AG'H'$; (b) $AH = EH'$; (c) $\angle AHF = \angle EH'F' = 90^\circ \Rightarrow AF = EF' \Rightarrow \triangle ABK \cong \triangle DEL$ because they are isosceles w/ equal height & base; Extend plane 3 and draw $\triangle CBD \cong \triangle ABK \cong \triangle DEL$; Given that the dihedral angle between $\triangle CBD$ and trapezoid ABDE = 180° by construction; the dihedral angle between $\square BDMN$ & $\square BKLD = 90^\circ$; the dihedral angle between trapezoid ABDE & $\square BKLD = \angle \epsilon$ by construction; \Rightarrow dihedral angle between $\triangle BCD$ and $\square BDMN$ $\angle \phi'' = \angle \phi$
 \therefore for $\angle BAK = \angle DEL = \angle BCD = \angle \theta$ ABCDE is flat Q.E.D

Introducing the Weaire Phelan approximation of minimum tension surfaces

The Polymorphic Elastegrity also yielded through folding the Weaire Phelan monododecahedron fig. 11(c). When arranged in the approximation of the minimum tension surface of bubbles, it leaves tetradecahedra fig. 11(d) as empty space in between fig. 11(e).

Several authors of this article participated in the 2019 Weaire Phelan workshop at ICERM organized by Glenn Whitney fig. 11(f).⁸ We connected edges of three lengths needed to assemble the Weaire Phelan matrix Fig. 11(g), which was an improvement by 0.3% in area^{9, 10} over Lord Kelvin's bubble approximation¹¹ Fig 11(b).

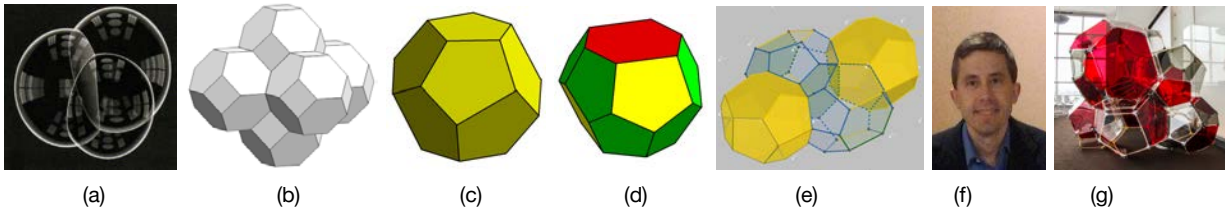


Fig. 11 Minimum surface tension (a) Bubbles; (b) Lord Kelvin's approximation, truncated octahedron 6 squares, 8 hexagons; (c) Weaire Phelan 1993 monododecahedron 4 equal sides and one longer, 106.6°, 102.6°, 121.6°, 106.6°, 102.6°; (d) Weaire Phelan tetradecahedron 1887: 4 pentagons congruent to the monododecahedral pentagons; 8 narrow pentagons, 107.02°, 107.02°, 101.54°, 112.21°, 112.21°; 2 hexagons with two parallel sides equal to the monodecahedral pentagon longer side, 126.87°, 116.57°, 116.57°, 126.87°, 116.57°, 116.57°; (e) 2 tetradecahedra & 2 monododecahedra; (f) Glenn Whitney; (g) WP ICERM 2019.

In an epiphany, during the workshop, it became clear that the Weaire Phelan monododecahedron lays along the polymorphic elastegrity path that had already been proven to exist between a rhombic dodecahedron and a cube. It could therefore be obtained through folding paper. Folding the flaps to the exact 121.59° and connecting the paper Weaire Phelan monopododecadra with wire and coffee stirrers would outline the WP tetradecahedra fig 12(a) in the space in between. Appropriate for G4G14 the tetradecahedron, is a polyhedron with **fourteen** faces, and was literally pulled out of thin air to present and report at the 2022 G4G.

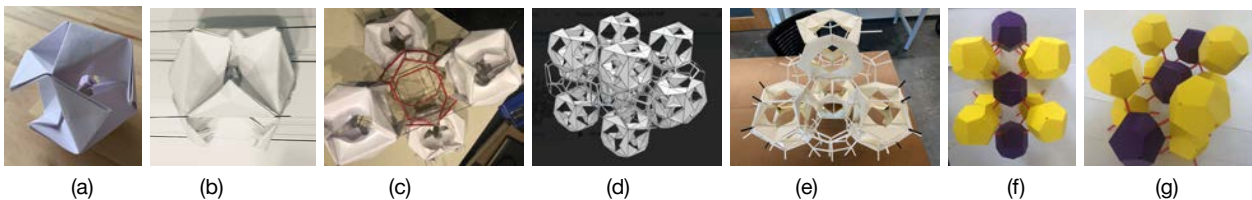


Fig. 12 (a) The polymorphic elastegrity folded as a monododecahedron, one angle 90°; (b) Paper folded WP monododecahedron 121.6°, 2X106.6°, 2X102.6°; (c) Study model of the tetradecahedron outlined with wire and coffee stirrers between paper folded WP monododecahedra; (d) Sketchup model of paper folded WP; (e) Paper model of WP monododecahedra, 3D printed connectors and straws; (f) and (g) 3D printed WP monododecahedra make the regularity of the WP pattern evident.

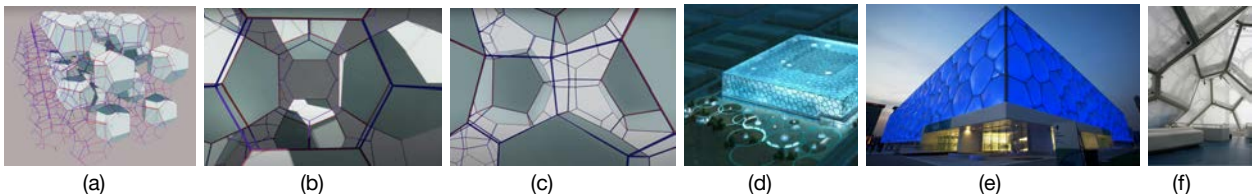


Fig. 13 (a) Digital flythrough WP showing regular staggered rows of monododecahedra; (b) View through a column of tetradecahedra showing alternating orientation of hexagons; (c) View through the narrow tetradecahedral pentagons; (d) Bird's eye view and (e) worm's eye view of the Beijing Olympics Aquatic Center; (f) Interior detail of the Beijing Olympics Aquatic Center.

This article started by citing Anni Albers's admonition to start designing by discovering simple rules and then using them to create complexity and interest. The Beijing Olympic Pool, also known as the "Water Cube", is an example of starting with the highly regular Weaire Phelan structure. The engineer Tristram Carfrae suggested it when the architect Chris Bosse, now of Laboratory of Visionary Architecture ([LAVA](#)) proposed a cube of bubbles for the Beijing Olympics Aquatic Center. The architect worked closely with the engineer to choose the plane

to cut through the regular pattern to create interest evoking irregular suds. Given that the structure had physical size, the section needed not to have nodes either just included or just excluded. They cut through the roof with two planes seven meters apart and through the walls three and a half meters apart. Underpinning the whimsical appearance of bubbles of the facade are the simple rules of the Weaire Phelan regular geometry.¹²

Epilogue

Polymorphic Elastegrity was discovered at the intersection of two experiments arising from the Bauhaus approach to design. We saw above how a failed experiment to create an easier assembly and more stable paper crystal in 1982 resulted in this new shape-shifting structure with interesting geometry. Beyond math a matrix of Polymorphic Elastegrity units exhibits -1 Negative Poisson's Ratio along the tetrahedral axes. Unrelated to the better known Poisson's distribution in statistics, Poisson's ratio is the ratio of lateral change over the axial resulting in response to an applied force. For example the ratio of how much a material expands laterally over how much it shortens when squeezed or how much it gets thinner over elongation under tension. Negative is the "perverse" material property as the New York Times called it,¹³ when a material gets smaller laterally when squeezed down and wider then pulled. The cooperative retraction of thirty-six elastic hinges suggests engineering applications for energy absorption. Since 2019 the Space Grant Opportunities in NASA STEM -NNX15AI06H has funded students with summer stipends to work on the Polymorphic Elastegrity. Physical and in-silico models (as engineers call animations) brought us closer to engineering applications in shock absorption similar to the [tensegrity NASA lander](#), mechanical analog sensors similar to [tensegrity sensors](#); that could withstand 800° on the Venus surface; and augments the [tensegrity conjecture in biology](#),¹⁴ opening avenues for discovering life on other planets. This work has been reported in the annual reports to NASA: Chris Norcross [2019](#), Kenneth Mendez [2020](#), Chelsy Luis [2021](#).



Fig. 14 Daanish Aleem Qureshi, [2022](#) recipient of NASA Scholar Summer Stipend working on a matrix of Polymorphic Elastegrities at the G4G14 offsite event, similar to the one he was funded to help with a sphere indentation experiment to measure auxeticity, which is a synonym for exhibiting Negative Poisson's Ratio or NPR, as is it is often abbreviated.

Endnotes

- ¹ E. Pavlides, T. Banchoff [Chiral Icosahedral Hinge Elastegrity's Shape-shifting](#) G4G12 Exchange Book pp 149, 2016
- ² E. Pavlides, P. Fauci [Chiral Icosahedral Hinge Elastegrity's Geometry of Motion](#), G4G13 Exchange Book pp 215, 2018
- ³ W. Lesser. [You Say to Brick: The Life of Louis Kahn](#), Farrar, Straus and Giroux, 2017
- ⁴ F. Horstman, [Preliminary Course and the Matière](#) *Josef Albers Papers*, Josef & Anni Albers Foundation, Bethany, CT.
- ⁵ Mitchell, D. [Mathematical Origami](#), Tarquin, 1997. Dave Mitchell named it "Skeletal Octahedron". In correspondence he identified Bob Neale as the earliest independent creator, who published it in 1968 as "Sixfold Ornament".
- ⁶ The maximum gate opening at dihedral angles 77.18° was derived from Alba Malaga's Geogebra Model visit [here](#).
- ⁷ [Syracusia](#), Wikipedia
- ⁸ Whitney, "Math's Bubbling (not!) Over." [video](#) of the structure <https://www.facebook.com/icerm/videos/567992044040173/>
- ⁹ R. Kusner and J. Sullivan, "Comparing the Weaire-Phelan equal-volume foam to Kelvin's foam," *Forma* 11:3, 1996, pp 233-242.
- ¹⁰ D. Weaire & R. Phelan, *A counterexample to Kelvin's conjecture on minimal surfaces*, *Phil. Mag. Lett.* v. 69, 1994, 107-110
- ¹¹ W. Thomson, Lord Kelvin, *On the division of space with minimum partitionial area*, *Phil. Mag.* vol. 24, 1887, 503.
- ¹² Correspondence with Arup fellow and engineer Tristram Carfrae and architect Chris Bosse, now of ([LAVA](#)).
- ¹³ James Gleick, *A Perverse Creation Of Science: Anti-Rubber*, *New York Times*, April 14, 1987.
- ¹⁴ D. Ingber. "Tensegrity as the architecture of life." IASS, Boston, USA, 2018, pp. 1-4.