G4G14 Gift
The Meta-Hilbert Curve $\rightarrow$ 14 Hilbert Curves

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Abstract

The Meta-Hilbert Curve construction is an infinite sequence C_n of space-filling curves, each a continuous mapping from the unit interval onto every point within a rotationally symmetric fractal tile, each the same area. The boundary of each C_n comprises eight "fractalized" Hilbert Curve approximation paths, four facing outwards, four inwards. The sequence of boundary fractal dimensions converges to 2.0, i.e., space-filling. C_∞ has twice the area of any C_n, but comprises 14 piecewise-connected Hilbert Curves, some coinciding with others.

The number 14 is not normally associated with any mathematical object demonstrating four-fold rotational symmetry in the plane. But due to combinatorial constraints arising from the need for open-ended, piecewise-connected continuity, and the interesting things that can sometimes happen when one takes a limit, the Meta-Hilbert Curve comprises $6 + 6 + 2 = 14$ Hilbert Curves. The second 6 lie, in reverse order, on top of the first 6.

The following is a condensed version of a much fuller exposition and motivation, with animations of the construction and interactive drawings, taken from my dynamic, electronic book, *Hilbert Curves* [1], currently published for iPad and iOS devices, and M1 MacOS computers, demoed at G4G14. The construction was first minimally described (under the name "Inside-Out Curve") in a paper on loops in DNA [2]. My collaboration with primary authors of that paper began with a chance meeting at the G4G10 Gathering for Gardner conference.

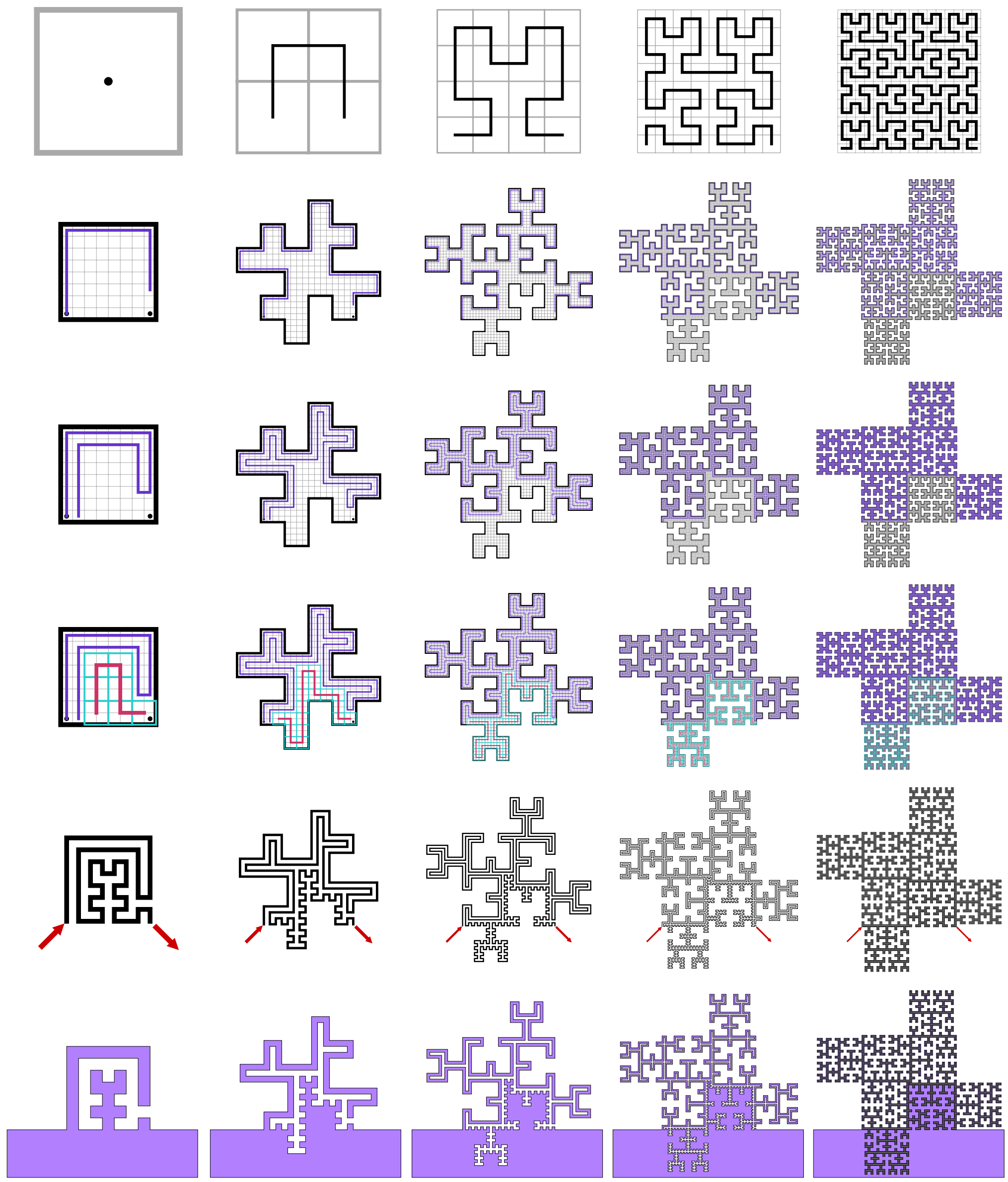

Figure 1: The Meta-Hilbert Curve Construction

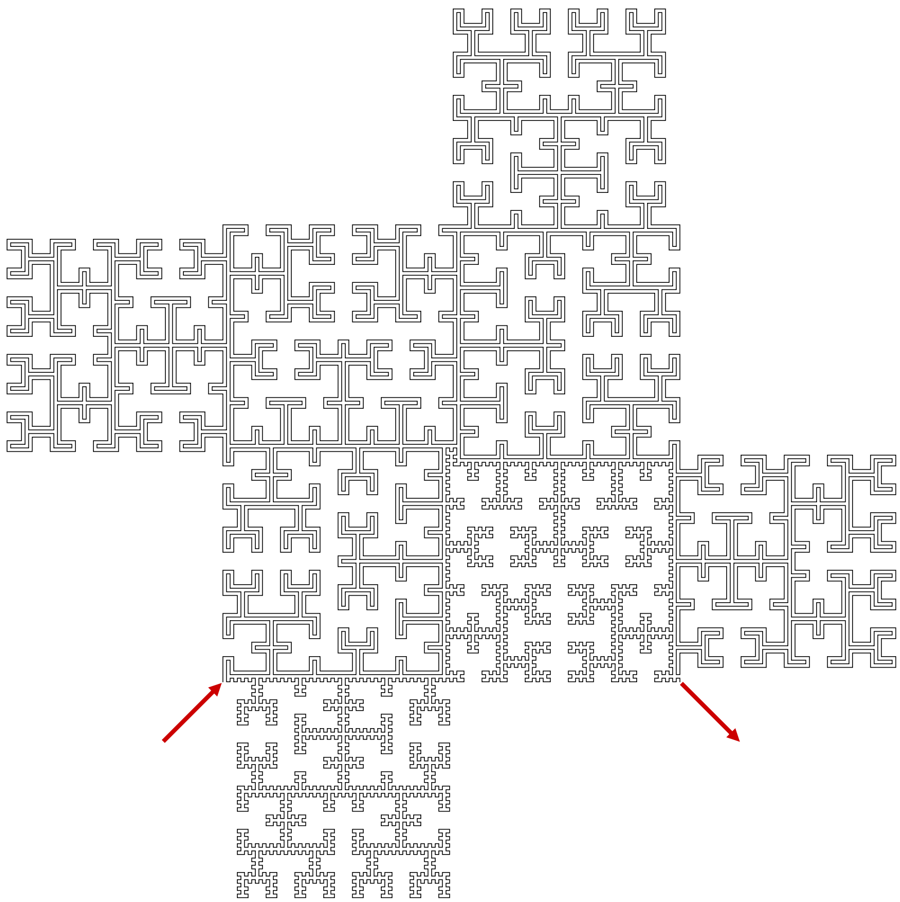

Figure 2: The Meta-Hilbert Curve motif for C_4 threads its prototile from a (left arrow) to b (right arrow) in such a way that it hews within ϵ (one sub-square width) of the outside or the inside of six Hilbert Curve approximation paths, two up (facing right, then left), two over (facing down, then up), and two down (facing left, then right). Within 2ϵ of b, the path turns around and hews within 2ϵ of the same six approximation paths, but in reverse order. Then, from within 2ϵ of a, the path completes its journey to b by hewing within 2ϵ along the bottom two Hilbert Curve approximation paths (facing down, then up), vibrating as a square-wave would. Scaled, rotated, and connected copies of this path would build the second approximation of this motif's self-similar, space-filling curve. The eventual fractal tile's boundary's dimension increases to 2.0 (with $\epsilon \to 0$) as increasingly detailed Hilbert Curve approximation paths are relied upon to build each fractal tile C_n. The square wave vibrations disappear in the limit, so the mapping to the final two Hilbert Curves has half the instantaneous "speed" of the previous 12 Hilbert Curves.

Approximation paths to the Hilbert Curve illustrate the sequence of always edge-adjacent sub-squares that increase in number as the square of the reciprocal of their decreasing-to-zero size. The sub-squares converge to the continuous Hilbert Curve, a fractal of dimension 2.0.

But the same is true of the "fractalized" boundaries of C_n. The connected area of each C_n becomes increasingly wispy, elongated, and branched, while the tile boundaries become increasingly close to space-filling Hilbert Curves. The interior area vanishes at the same limiting "moment" that the points to which the boundary is converging from both inside and outside take over being the area of a rotationally symmetric tile (with linear boundary). So if $A(n)$ is the normalized unit area of every C_n, we have $A(n) = 1$ for all finite n, but $\lim_{n\to\infty} A(n) = 2$.

The foregoing Meta-Hilbert construction works using any order-n approximation path for any square-filling generalization of the Hilbert Curve based on the $n \times n$ recursive subdivision of the square; see, e.g., Figure 102 in [1], which uses an order-3 Wunderlich Curve approximation path to build an element of a Meta-Wunderlich Curve sequence of space-filling curves.

References

[1] McKenna, D. M., *Hilbert Curves: Outside-In and Inside-Gone*, Mathemaesthetics, Inc. (2019), ch. 6. ISBN: 978-1-7332188-0-1 (iPad/iPhone/MacOS eBook available at `https://apps.apple.com/us/app/hilbert-curves/id1453611170` (as of 4/14/2022).

[2] Sanborn, A. L., Rao, S. S. P. , et al, "Chromatin extrusion explains key features of loop and domain formation in wild-type and engineered genomes," Proceedings of the National Academy of Sciences, **112**, p. 47 and Fig. S3 (Nov. 24, 2015). See `https://www.pnas.org/doi/10.1073/pnas.1518552112` (as of 4/14/2022).

[3] Sloane, Neil A. J., *The On-Line Encyclopedia of Integer Sequences*, accessible electronically at `http://www.oeis.org/search?q=A000532` (as of 4/14/2022).