Generalization of Cone-pass and Continued Fraction
Cone-puter

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1. INTRODUTION

In my previous paper, "Cone-pass" [1], I presented a method for constructing the golden angle by manipulating a circular sheet with slits to create a cone. (Figure 1)

In addition to the "central angle *w*", which is a parameter that determines a cone shape, I brought in the concept of "overlap angle *w'* ". (Figure 2)

In order to consider both the central angle *w* and the overlap angle *w'* as real numbers in the $[0,1]$ interval, let's assume that the circumference is *1*, and thus we are dealing with a circular sheet of radius $r = 1/(2\pi)$. The circular sheet is an ideal paper with zero thickness, so no matter how many sheets are stacked on top of each other, there is no increase in thickness.

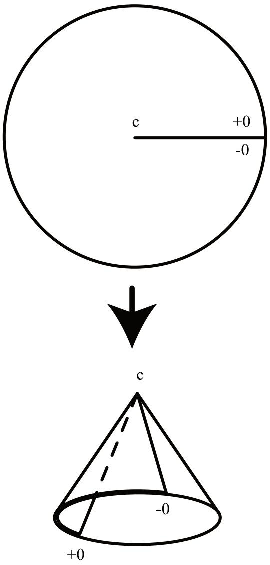


Fig.1
Make a cone from a circular sheet with a slit in it.

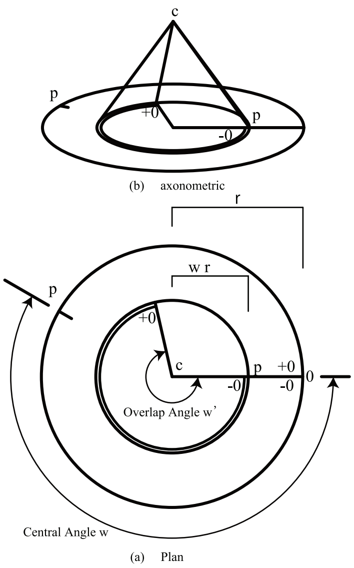


Fig. 2
Central Angle and Overlap Angle

The central angle and the overlap angle generally do not coincide, as shown in Figure 2. However, if we continue recursively by finding the central angle from a cone with an appropriate overlap angle, creating a cone with the next overlap angle, and then finding a updated central angle, both the central angle and the overlap angle converge to the golden angle $(\tau-1) = (-1+\sqrt{5})/2 \approx 222.5°$. I named it the "recursive cone method" and showed in a previous paper that it corresponds exactly to the continued fraction expansion of the golden ratio. [1]

The cone with the golden angle as its central angle was called the "golden cone," and its elevation was composed of a right triangle with $r : r/\tau : \sqrt{(r/\tau)}$. (Figure 3, left)

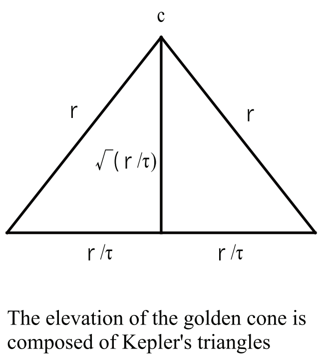
As I noticed after the presentation of the previous paper, this triangle was the same as the so-called "Kepler's triangle" of $\tau : 1 : \sqrt{\tau}$. Kepler said, "There are two treasures in geometry. One is the Pythagorean theorem and the other is the golden ratio. The first may be compared to a gold nugget, and the second may be called a precious jewel.” This triangle is truly a combination of gold and jewels.

Kepler's triangle can also be seen in the elevation of the Pyramids of Giza (Fig. 3, middle). For a long time, I did not think this fact was very important. Since a right triangle with $\tau : 1 : \sqrt{\tau}$ can be easily constructed with a ruler and compass, it is no wonder that it appears everywhere in architecture. It's like if you draw a circle with a compass, you'll find the transcendental number π there.

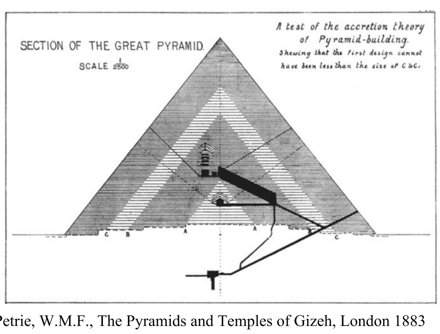
However, as I myself wrote in my previous paper, "This triangle is fascinating and many other functions are likely to be discovered," I now believe that the use of Kepler's triangle in the pyramids was probably not an accident. It is a fact that this right triangle was constructed by the Egyptians thousands of years before Kepler, and the Egyptians of that time must have regarded this triangle as special. In fact, it may have had some function in the construction and mechanics of the pyramid, just as the geometric function of the golden angle was found to be constructed from Kepler's triangle.

In this paper, I attempt to generalize the recursive cone method. The question is as follows.

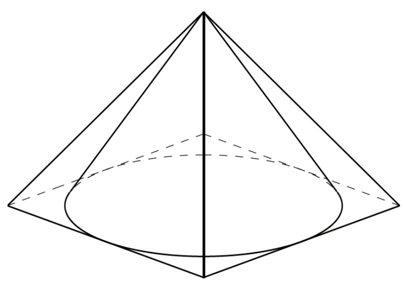
What angles other than the golden angle can be determined by the recursive cone method?



The elevation of the golden cone is composed of Kepler's triangles



Petrie, W.M.F., The Pyramids and Temples of Gizeh, London 1883



The golden cone inscribed in the pyramid

Fig. 3
Pyramid and Kepler's triangle

2. Generalization of the cone method

In the previous paper, we considered only the case where the paper overlaps twice when making a cone from a circular sheet with a slit in it. (Figure 2)
In other words, we did not consider the case where the central angle *w* of the cone is less than *1/2*. Let's lift this limitation and consider the problem.

Consider a circular sheet of circumference *1*, made of ideal paper of thickness *0*. The radius of the circle is $r = 1/(2\pi)$. As shown in Fig. 4, the cut of the radius is set as the base point *0*, and the cut edges are set as *+0* and *-0*, respectively.
The circular paper is on a horizontal plane, which we call the surface plate, and an ideal axis of zero thickness is assumed to stand vertically at the center *c*.
The central angle *w* of the cone is defined to be *[0, 1]*. In other words, it can be any real number with $0 \leq w \leq 1$.
Choose an appropriate central angle w as shown in Fig. 4, and set the point of the circumference as P_1. Keeping the *-0* end of the slit aligned on the base point *c0*, slide the *+0* end clockwise to make a cone with P_1 and *-0* ends matched. (Figures 5 and 6)
Here, the reciprocal of the central angle *w*, *1/w*, is a value that should be called "overlap degree," meaning how much the papers overlap.
The overlap degree can be a real number with $1 \leq 1/w \leq \infty$.
The base radius of the cone made by the central angle *w* is *wr*.
When the central angle $w = 1$, i.e., the overlap degree is *1*, the cone is perfectly flat.
When the central angle *w* approaches zero infinitely, it becomes a cone wound infinite times and degenerates into a line segment of length *r*.

Let "*a*" be the integer part of *1/w*. *a* means the number of times that the *+0* end of the cone meets the *-0* end when it is slid to form a cone, so let's call it the "conjunction number *a*" (Fig. 5). The angle formed by the *(a+1)* overlapping pieces is called the overlap angle *w'* (Fig. 5).
The overlap angle *w'* is defined as follows

$$w'=(1-aw)/w \quad \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

Then, mark all the cone surfaces of *a* that match the edge of *-0* with $P_2, P_3, \ldots, P_a$ (Figure 5, Figure 6)
Open the cone and return it to the circular sheet. (Figure 7)
The markings made are lined up *a* times with central angles *w* from *-0* end..
It is obvious that the last remainder up to *+0* represents the fractional part of *1/w*.
The angles formed by the *a* marks and *-0* are, as follows in turn,

1st central angle $w(1)=w$
2nd central angle $w(2)=2w$
・
・
a th central angle $w(a)=aw$

Let's denote the *b*th central angle as $w(b)=bw$, where *b* is an integer with $1 \leq b \leq a$.

Let's call the remaining angle that is less than the central angle the "fraction angle." The value of a fractional angle is *1-aw*.
Note that Figures 4, 5, 6, and 7 are examples of $w=3/11$.
In Figures 5, 6, and 7, the cross section of the cone is drawn in a spiral shape for convenience, but as mentioned at the beginning, the cone is composed of ideal paper of zero thickness, so the thickness of the cone surface is zero no matter how many times it is wound.

We are now ready for consideration.

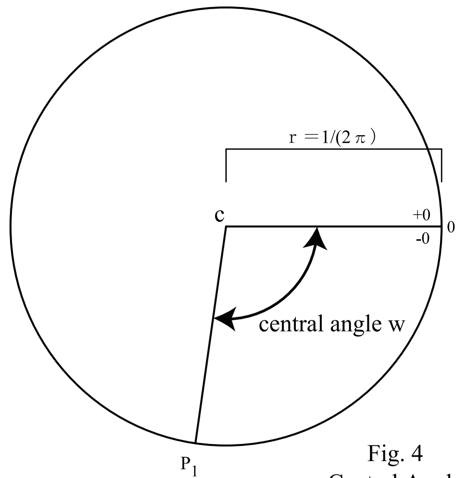


Fig. 4
Central Angle

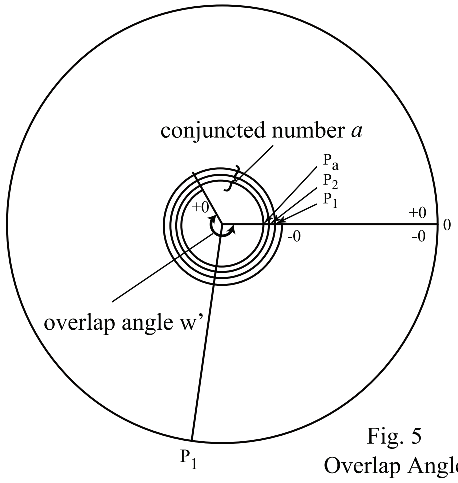


Fig. 5
Overlap Angle

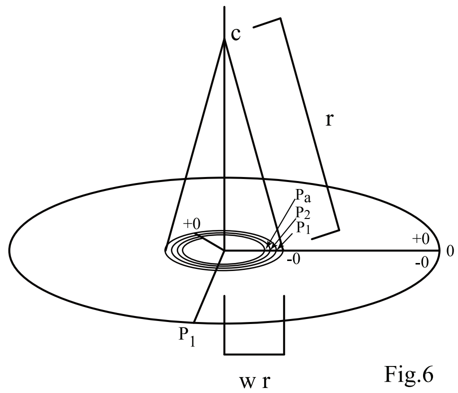


Fig.6
Axonometric of Cone

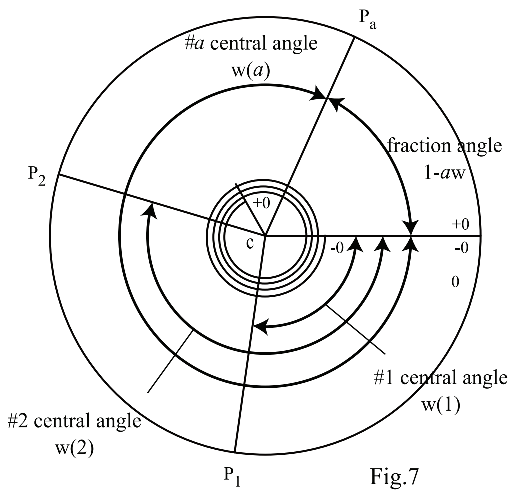


Fig.7
from #1 to #*a* Central Angles

3. Construct an arbitrary regular *N*-gon / arbitrary rational angle

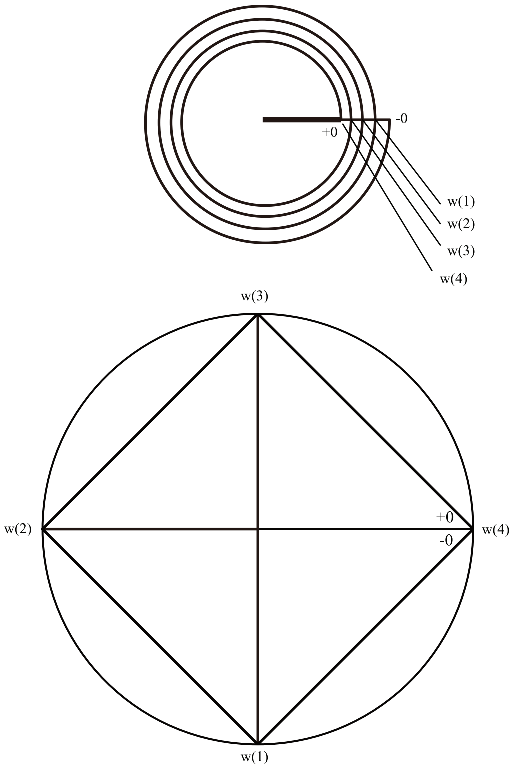


Fig. 8
Construction of a regular *N*-gon

Let us first consider the special case where the overlap degree $1/w$ is exactly integer N, as shown in Figure 8 above. The fraction angle $(1-aw) = 0$, and -0 and $+0$ make conjunction again. $a=N$ and the overlap angle $w' = 0$. As an example, Figure 8 on the left shows the case where $w(1)=1/4$ and $1/w=N=a=4$.
Mark all of the cone surfaces that match with the -0 end, return to the circular sheet, and connect the marked points on the circumference with lines in order to form a regular *N*-gon (Fig. 8, bottom).This is also self-evident. In other words, if you allow the cone method as a constructing method, you can construct any regular *N*-gon in a single operation.
Naturally, angles of rational numbers with N as the denominator can also be constructed. To find the angle of a rational number b/N, make a cone with exactly N overlap degree, and mark the bth central angle $w(b)$.

You can construct a regular pentagon with just a ruler and compass, but the next constructable regular polygons will have to wait until a regular *17*-gon. The fact was discovered by Gauss.
Later, it was known that “among regular *N*-gons where *N* is prime, such a construction is possible only if *N* is Fermat prime.”
In addition, Gauss showed that “a necessary and sufficient condition for a regular *N*-gon to be constructable is in the form of a product of Fermat primes differing in power from *N* by *2*.”
You can't even construct a regular *9*-gon with a ruler and compass.
Origami allows trisecting the corners, so you can fold a regular *9*-gon exactly. But even origami cannot fold arbitrary regular polygons.

4. Constructing irrational angles with the general recursive cone method: $\{a,b\}$ operation

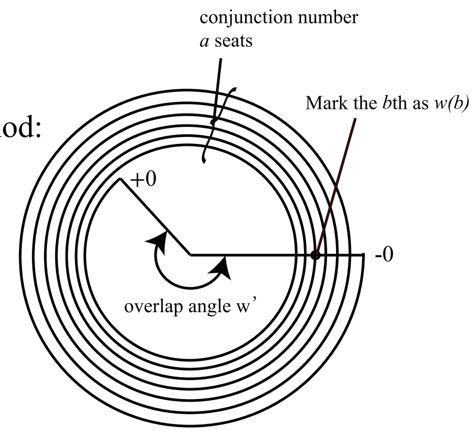


Fig. 9
General Recursive Cone Method
Schematic diagram of the $\{a,b\}$ operation
Example of $\{6,4\}$ operation

Next, let's consider the case where the overlap degree 1/w is not an integer.
$1/w > a$, resulting in an overlap angle. (Figure 9)
The equation to derive the central angle from the overlap angle is the inverse function of equation (1) .

$w=1/(a+w')$ ・・・・・・・・・・・・・(2)

This is the so-called *1*st central angle $w(1)$.
Therefore, after the second central angle are as folows.

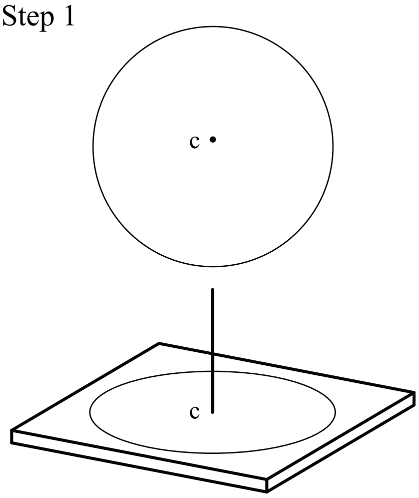
*2*nd central angle $w(2)=2/(a+w')$
*3*rd central angle $w(3)=3/(a+w')$
・
・
*a*th central angle $w(a)=a/(a+w')$

In general, the *b*th central angle is expressed as $w(b)=b/(a+w')$. (where *b* is an integer. $1 \le b \le a$)
Suppose that we now have a cone with an appropriate overlap angle w' and its conjunction number is *a*.(Figule 9)
１）Mark the *b*th cone surface matched with *-0*.
２）When opened in a circle, the angle between the marked point and -0 is the *b*th central angle. Mark the angle on the surface plate.
３）Raise the cone of conjunction number *a* again, and create a cone of overlap angle that match with the *b*th central angle. As before, make a new mark on the the *b*th cone surface matched with *-0*.
４）When opened in a circle, the angle between the marked point and -0 is the updated *b*th central angle. Mark the angle on the surface plate.
５）Return to the operation in 3) above, and repeat indefinitely thereafter.

In any initial form, if the above recursive operation with integer parameters *a* and *b* is repeated, the central angle and overlap angle will converge to the same angle. Let's call this recursive operation the "$\{a,b\}$ operation".
In a previous paper [1] I showed a chart of the recursive cone method for constructing the golden angle. The next page shows the procedure (recipe) for the $\{a,b\}$ operation, updated as a general recursive cone method.

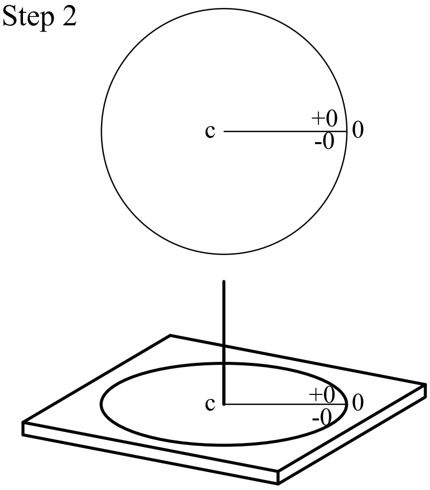
General Recursive Cone Method
Procedure
$\{a,b\}$ operation

Step 1



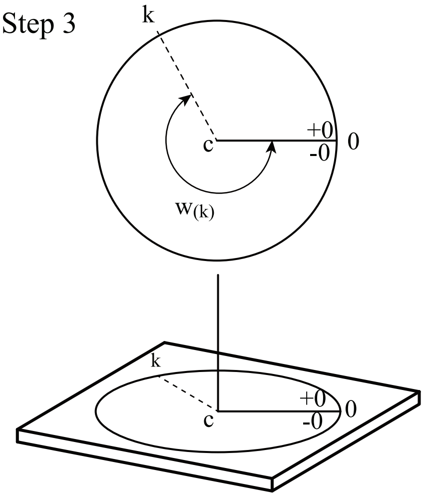
First, prepare a surface plate with a needle that stands vertically.
The thickness of the needle is ideally considered to be zero.
The position of the needle is c, and a circle of radius $r=1/(2\pi)$ is drawn on the surface plate with c as the center.
Since the height of the cone can never be longer than the radius of r, the length of the needle should be at least r.
The top figure is a plan view and the bottom figure is an axonometric.

Step 2



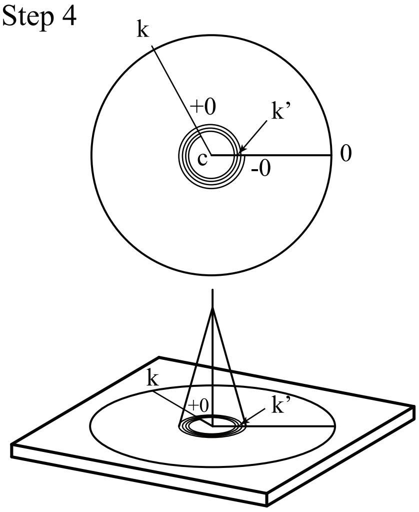
Cut out a circle of radius r from an ideal paper of thickness 0, and make a single slit from the center c toward the circumference.Thread the needle through the center of the paper circle and lay it on the surface plate.
Mark the base point 0 on the circumference of the surface plate located at the edge of the slit.
The edge point belonging to the upper part is $+0$, and the edge point belonging to the lower part is -0. The base point on the surface plate is assumed to be an unsigned 0.

Step 3



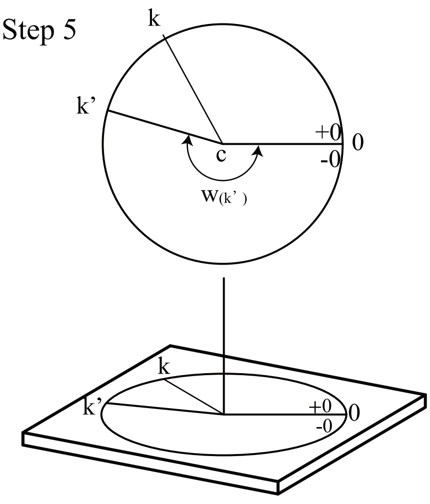
Mark "k" at an appropriate position on the circumference of the surface plate.
k is an integer. ($k \geq 1$)
The angle from the base point 0 to k in a clockwise direction is denoted as $w_{(k)}$. ($0 \leqq w_{(k)} \leqq 1$)

Step 4



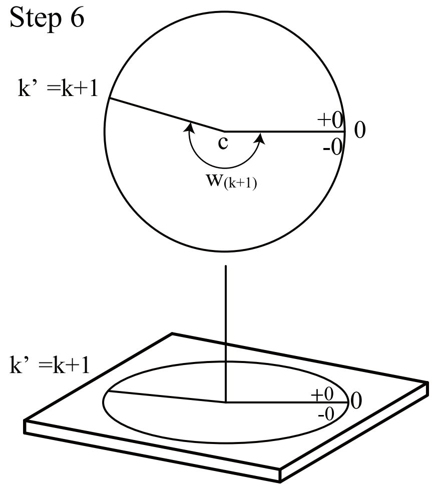
Draw a straight line ck from the center c of the circle on the surface plate to k.
Keeping the -0 end of the paper circle aligned on the base point $c0$, slide the $+0$ end clockwise and make a cone such that the $+0$ end lies on the line ck after winding a times.
This means that the overlap angle w' is set to the same angle as central angle $w(k)$.
Then mark k' on the bth cone surface matched with the -0 end.

Step 5



Return the cone to a flat surface again and align the slit with the base point 0.
The angle $w_{(k')}$ from starting point 0 clockwise to k' is as follows.
$w_{(k')}=b/(a+w_{(k)})$
This is the bth central angle of the cone.

Step 6



k' is set to $k+1$, then k in Step 3 is updated to the angle of $k+1$, and the same operation is repeated infinitely.

$w_{(\infty)}$ converges to $\frac{-a+\sqrt{a^2+4b}}{2}$

5. $\{a,b\}$ operations and continued fractions

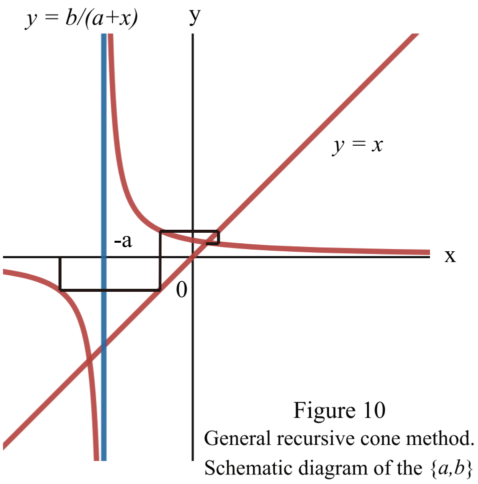


Figure 10
General recursive cone method.
Schematic diagram of the $\{a,b\}$

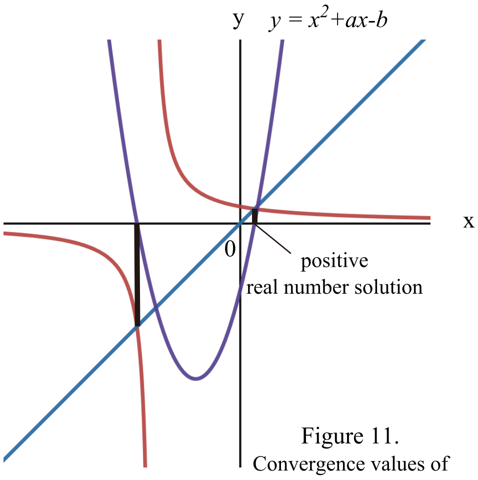


Figure 11.
Convergence values of
$\{a,b\}$ operations

The convergence of the $\{a,b\}$ operation with the recursive cone method can be understood from the hyperbola of $y=b/(x+a)$ and the line of $y=x$.The convergence can be visualized as a trajectory of recursive operations, where $y=b/(x+a)$ is obtained from the appropriate value of x, and then $y=b/(x+a)$ is updated with the new x.(Figure 10)
The value that converges is the positive real solution of $x^2+ax-b=0$. (Figure 11)
Therefore, if we repeat the $\{a,b\}$ operation infinitely, the convergence value is as follows.

$$w = w' = \frac{-a+\sqrt{a^2+4b}}{2} \quad \cdots\cdots (3)$$

The convergence angle list for the conjunction number a is 1, 2, 3,. .7 is shown in Table 1 on the next page.
In particular, when a is even, the central angle w = overlapping angle w' converged by the recursive cone method can be shown to cover the fractional part of the square root of a non-square natural number.
The value of $a/2$ is the integer part N of the square root of a natural number.

$$a=2N \quad \cdots\cdots (4)$$

(The case where the natural numbers are square numbers corresponds to the case where $1/w$ is exactly an integer in Section 3.)
Instead of relying on the list in Table 1, let's construct the angle corresponding to the fractional part of the square root of a natural number n that is not a square number.
First, we figure the integer part N of square root of the natural number n.
From equation (4), since $2N$ is the value of the conjunction number a, make a cone winding circular sheets as many times as a.
The value of b is as follows

$$b = n - N^2 \quad \cdots\cdots (5)$$

By using the general recursive cone method$\{a,b\}$ operation formulated in Section 4, we can construct the angle that is the fractional part of $\sqrt{n}$.
The recursive cone method $\{a,b\}$ operation is equivalent to recursively computing the following recurrence formula.

$$x_{k+1}=b/(a+x_k) \quad \cdots\cdots (6)$$

From the form of the recurrence formula (6), we can see that it can be expressed as a continued fractions with the partial denominator constant in a and the partial numerator constant in b. (Figure 12)
Following the reference [2], when the partial numerator of a continued fraction is a constant b, it is called a "b-continued fraction" and is denoted by $[C_1,C_2,C_3,\ldots]_b$. In particular, when the partial numerator is always 1, it is denoted by $[C_1,C_2,C_3,\ldots]_1$. This is the so-called "regular continued fraction".
The recursive cone method is a b-continued fraction, and in particular, the partial denominator is also a constant a, so it can be written minimally as $[\bar{a}]_b$. The upper line of $\bar{a}$ denotes that a repeats infinitely. The author thinks that this type of continuous fraction notation is more essential than regular continuous fractions. Only two integers are enough to informatize about an irrational number, and the convergence is fast. If I were a genetic designer, I would choose this over regular continued fractions.

The $\{a,b\}$ operation, or the continued fraction $[\bar{a}]_b$, can be used to calculate the convergents of the $\sqrt{n}$ you want to find, as desired. Let's find $\sqrt{83}$ as an example.
We can know by rote that the integer part N of $\sqrt{83}$ is 9. From equations (4) and (5), we get

$$a = 2N = 2*9 = 18$$
$$b = n - N^2 = 83 - 9^2 = 2$$

So if we calculate the continued fraction $[\overline{18}]_2$, we get the fractional part of $\sqrt{83}$, that is $-9+\sqrt{83}$.
The recurrence formula is as follows.

$$x_{k+1}=2/(18+x_k)$$

If we start with the initial value $x_0=0$,

$$x_1=2/(18+0)=1/9$$
$$x_2=2/(18+1/9)=18/163$$
$$x_3=2/(18+18/163)=163/1476$$

and a fairly good convergent can be obtained with only three recursive operations.
And similarly, the angle of $163/1476$ can be constructed exactly by the general recursive cone method $\{18,2\}$operation.

$$x_\infty = \cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cdots}}}}$$

Figure 12.
$\{a,b\}$ operation and equivalence of $[\bar{a}]_b$ continued fractions

Conjunction Number a	#b Central Angle b	The solution of $x^2+ax-b=0$ $\frac{-a+\sqrt{a^2+4b}}{2}$ Convergence Angle	Recurrence Formula $w'_{k+1}=\frac{b}{a+W'_k}$	$[\overline{a}]_b$ Continued Fraction $\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cdots}}}$	Regular Continued Fraction $\cfrac{1}{C_1+\cfrac{1}{C_2+\cfrac{1}{C_3+\cdots}}}$	Approximate Value	Approximability
1	1	$\frac{-1+\sqrt{5}}{2}$	$w'_{k+1}=\frac{1}{1+W'_k}$	$[\overline{1}]_1$	$[\overline{1}]_1$	0.618033989	16
2	1	$-1+\sqrt{2}$	$w'_{k+1}=\frac{1}{2+W'_k}$	$[\overline{2}]_1$	$[\overline{2}]_1$	0.414213562	9
2	2	$-1+\sqrt{3}$	$w'_{k+1}=\frac{2}{2+W'_k}$	$[\overline{2}]_2$	$[\overline{1,2}]_1$	0.732050808	12
3	1	$\frac{-3+\sqrt{13}}{2}$	$w'_{k+1}=\frac{1}{3+W'_k}$	$[\overline{3}]_1$	$[\overline{3}]_1$	0.302775638	6
3	2	$\frac{-3+\sqrt{17}}{2}$	$w'_{k+1}=\frac{2}{3+W'_k}$	$[\overline{3}]_2$	$[\overline{1,1,3}]_1$	0.561552813	11
3	3	$\frac{-3+\sqrt{21}}{2}$	$w'_{k+1}=\frac{3}{3+W'_k}$	$[\overline{3}]_3$	$[\overline{1,3}]_1$	0.791287847	10
4	1	$-2+\sqrt{5}$	$w'_{k+1}=\frac{1}{4+W'_k}$	$[\overline{4}]_1$	$[\overline{4}]_1$	0.236067977	5
4	2	$-2+\sqrt{6}$	$w'_{k+1}=\frac{2}{4+W'_k}$	$[\overline{4}]_2$	$[\overline{2,4}]_1$	0.449489743	7
4	3	$-2+\sqrt{7}$	$w'_{k+1}=\frac{3}{4+W'_k}$	$[\overline{4}]_3$	$[\overline{1,1,1,4}]_1$	0.645751311	11
4	4	$-2+\sqrt{8}$	$w'_{k+1}=\frac{4}{4+W'_k}$	$[\overline{4}]_4$	$[\overline{1,4}]_1$	0.828427125	9
5	1	$\frac{-5+\sqrt{29}}{2}$	$w'_{k+1}=\frac{1}{5+W'_k}$	$[\overline{5}]_1$	$[\overline{5}]_1$	0.192582404	4
5	2	$\frac{-5+\sqrt{33}}{2}$	$w'_{k+1}=\frac{2}{5+W'_k}$	$[\overline{5}]_2$	$[\overline{2,1,2,5}]_1$	0.372281323	8
5	3	$\frac{-5+\sqrt{37}}{2}$	$w'_{k+1}=\frac{3}{5+W'_k}$	$[\overline{5}]_3$	$[\overline{1,1,5}]_1$	0.541381265	10
5	4	$\frac{-5+\sqrt{41}}{2}$	$w'_{k+1}=\frac{4}{5+W'_k}$	$[\overline{5}]_4$	$[\overline{1,2,2,1,5}]_1$	0.701562119	9
5	5	$\frac{-5+\sqrt{45}}{2}$	$w'_{k+1}=\frac{5}{5+W'_k}$	$[\overline{5}]_5$	$[\overline{1,5}]_1$	0.854101966	8
6	1	$-3+\sqrt{10}$	$w'_{k+1}=\frac{1}{6+W'_k}$	$[\overline{6}]_1$	$[\overline{6}]_1$	0.16227766	4
6	2	$-3+\sqrt{11}$	$w'_{k+1}=\frac{2}{6+W'_k}$	$[\overline{6}]_2$	$[\overline{3,6}]_1$	0.31662479	5
6	3	$-3+\sqrt{12}$	$w'_{k+1}=\frac{3}{6+W'_k}$	$[\overline{6}]_3$	$[\overline{2,6}]_1$	0.464101615	6
6	4	$-3+\sqrt{13}$	$w'_{k+1}=\frac{4}{6+W'_k}$	$[\overline{6}]_4$	$[\overline{1,1,1,1,6}]_1$	0.605551275	11
6	5	$-3+\sqrt{14}$	$w'_{k+1}=\frac{5}{6+W'_k}$	$[\overline{6}]_5$	$[\overline{1,2,1,6}]_1$	0.741657387	10
6	6	$-3+\sqrt{15}$	$w'_{k+1}=\frac{6}{6+W'_k}$	$[\overline{6}]_6$	$[\overline{1,6}]_1$	0.872983346	7
7	1	$\frac{-7+\sqrt{53}}{2}$	$w'_{k+1}=\frac{1}{7+W'_k}$	$[\overline{7}]_1$	$[\overline{7}]_1$	0.140054945	4
7	2	$\frac{-7+\sqrt{57}}{2}$	$w'_{k+1}=\frac{2}{7+W'_k}$	$[\overline{7}]_2$	$[\overline{3,1,1,1,3,7}]_1$	0.274917218	9
7	3	$\frac{-7+\sqrt{61}}{2}$	$w'_{k+1}=\frac{3}{7+W'_k}$	$[7]_3$	$[\overline{2,2,7}]_1$	0.405124838	6
7	4	$\frac{-7+\sqrt{65}}{2}$	$w'_{k+1}=\frac{4}{7+W'_k}$	$[\overline{7}]_4$	$[\overline{1,1,7}]_1$	0.531128874	8
7	5	$\frac{-7+\sqrt{69}}{2}$	$w'_{k+1}=\frac{5}{7+W'_k}$	$[\overline{7}]_5$	$[\overline{1,1,1,7}]_1$	0.653311931	10
7	6	$\frac{-7+\sqrt{73}}{2}$	$w'_{k+1}=\frac{6}{7+W'_k}$	$[\overline{7}]_6$	$[\overline{1,3,2,1,1,2,3,1,7}]_1$	0.772001873	9
7	7	$\frac{-7+\sqrt{77}}{2}$	$w'_{k+1}=\frac{7}{7+W'_k}$	$[\overline{7}]_7$	$[\overline{1,7}]_1$	0.887482194	7

Table 1

List of recursive cone $\{a,b\}$ operations

The "Approximability" in the rightmost column is a comparison with the difficulty of approximating the Golden angle (*τ-1*). It is the order of the convergent in the regular continued fraction expansion of each convergence angle, which is comparable to the approximation accuracy of the *16*th convergent in the regular continued fraction expansion of the Golden angle.

6. $[\overline{a}]_b$ Continued Fraction and Regular Continued Fraction

The above recursive operations are easy to write as a computer program.
However, it is the focus of this paper to obtain an equivalent solution as the central angle of a cone rather than a computer. Cone -pass or should I call it Cone-puter? This idea did not come from the knowledge of continued fractions, but from the consideration of the geometric figure of a cone, which naturally led to the continued fraction expansion.
The simplicity of $[\overline{a}]_b$ contiued fraction is the simplicity of cone. Its convergence is more than that of the regular continuous fraction $[C_1,C_2,C_3,...]_1$.
Nevertheless, regular cotinued fractions, which are the mainstream in the study of continued fractions, have the great advantage of outputting the best convergents that are already irreducible, even though the procedure is somewhat more complicated.
I can't find a good way to convert a b-continued fraction with $b>1$ into a regular continued fraction. Wikipedia "Generalized Continued Fraction" shows how to convert the partial numerator to 1, but in that case, the partial denominator is not always an integer, so it is not a conversion to a regular continued fraction.
In the end, a good way to convert $[\overline{a}]_b$ continued fractions into regular continued fraction would be to start with an initial value of x_0 of 0 in the recurrence formula in (6), repeat the recursive operation several times to make convergents of enough high order, and then use Euclidean algorithm to make regular continued fractions again. There is no need to proceed with infinite Euclidean algorithm. If the value of the conjunctuion number a appears in the partial denominator, the regular continued fraction is fixed. The reason for this is that the sequence of partial denominators until the appearance of the conjunction number a repeats itself thereafter.
As a side note, the sequence of partial denominators that precede a is symmetrical (palindromic) [3]. This can be observed in Table 1. This phenomenon is also interesting and awaits geometrical clarification.
Let's illustrate the regular continued fractional transformation with the aforementioned $-9+\sqrt{83}$.
The first three of the convergents output in $[\overline{18}]_2$ are

The 1st convergent is $1/9$
The 2nd convergent is $18/163$
The 3rd convergent is $163/1476$

Using Euclidean algorithm to expand $163/1476$ into a regular continued fraction is $[9,18,9]_1$.
Therefore, the regular continued fraction of $-9+\sqrt{83}$ is $[\overline{9,18}]_1$. The first three convergents output from this regular continued fractions are as follows.

The 1st convergent is $1/9$
The 2nd convergent is $18/163$
The 3rd convergent is $163/1476$

This is consistent with that of $[\overline{18}]_2$.
However, the convergents output by a general $[\overline{a}]_b$ continued fraction are not always consistent with the convergents output by the regular continued fraction converted by the above method.
For example, the case of $\sqrt{7}$ is remarkable. As shown in Table 2, when comparing $[\overline{4}]_3$ and $[\overline{1,1,1,4}]_1$ in a continued fraction of $-2+\sqrt{7}$, none of the convergents match and convergence is much faster for the $[\overline{4}]_3$ continued fraction.

$-2+\sqrt{7}=0.645751311..$

$[\overline{4}]_3$ continued fraction	Value	Error	$[\overline{1,1,1,4}]_1$ Continued Fraction	Value	Error
3/4	0.75	0.1042486889	1/1	1	0.3542486889
12/19	0.631578947	−0.0141723637	1/2	0.5	−0.1457513111
57/88	0.647727273	0.0019759617	2/3	0.666666667	0.0209153556
264/409	0.645476773	−0.0002745384	9/14	0.642857143	−0.0028941682
299/463	0.645789474	0.0000381626	11/17	0.647058824	0.0013075125
463/717	0.645746007	−0.0000053045	20/31	0.64516129	−0.0005900207
494/765	0.645752048	0.0000007373	31/48	0.645833333	0.0000820223

Table 2
Comparison of $[\overline{a}]_b$ continued fraction and regular continued fraction $[\overline{C_1,C_2, ... ,a}]_1$ of $-2+\sqrt{7}$

To be able to use the cone method to construct the angle of a convergent calculated from a regular continued fraction. Suppose that its convergent q/p is represented by $[C_1 , C_2 , ... , C_k]_1$. The $\{a,b\}$ operation described above was assumed to be repeated infinitely, but in the case of a finite number of operations, let's denote it by $\{a , b\}_i$ with the number of times i. Then the convergent q/p, represented by a continued fractions in $[C_1 , C_2 , ... , Ck]_1$, can be constructed by the following sequence of $\{a , b\}_i$ operations.

$\{C_k , 1\}_1 \quad \{C_{k-1} , 1\}_1 \quad ... \quad \{C_2 , 1\}_1 \quad \{C_1 , 1\}_1$

For example, the fourth convergent $9/14$ in $-2+\sqrt{7}$ can be constructed using the sequence $\{4, 1\}_1 \quad \{1, 1\}_3$.
Moreover, even for any convergent of regular continuous fraction expansions without circulating partial denominators, such as transcendental numbers, the corresponding angles can be constructed by above cone method.
As noted in Section 3, the rational number q/p can be constructed directly from a cone with exactly p-fold overlap, but it is also valid to use regular continued fractions as above. This is because, although the number of operations is increased, the number of windings is reduced and the accuracy of the construction is practically improved.

7. Odd conjunction number *a*

Now let's consider the case where the integer part *a* of *1/w*, i.e. the conjunction number *a* is odd.
At the least conjunction number *a=1*, the central angle is only *w(1)=w*, and the angle of convergence of the {*1,1*} operation, i.e. the limit of the $[\overline{1}]_1$ continued fraction, is none other than the recursive cone method of the golden angle discussed in the previous paper [1]. That is, the golden angle is the convergence angle of the simplest recursive cone method and the convergence value of the simplest $[\overline{a}]_b$ continued fraction.

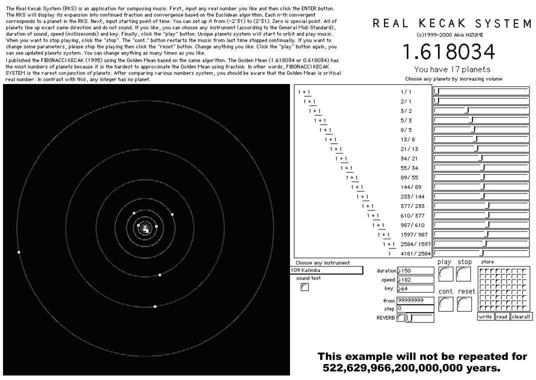


Fig. 14
Real Kecak System (2000)

If the conjunction number *a* shown in Table 1 is an odd number of 3 or more, the similar geometrical function as the golden angle is expected.
I've been having fun creating music that reflects its continued fractional structure. The application software Real Kecak System [4], developed in 2000, generates poly-rhythmic music that reflects the continuous fractional structure of any real number.
So far, apart from the golden ratio, I have explored the variety of quasi-periodic poly-rhythms by using the fractional part of the square root of integers, especially prime numbers, all of which have an even conjunction number *a*. Now I feel like I have discovered a new continent of poly-rhythmic music with odd conjunction numbers. The list in Table 1 seems to be a ranking of music in order of popularity. In first place, of course, is the music of the Golden Angle.

8. Constructing of line segments and list of Pisot numbers

The angle of the square root of a natural number *n* could be constructed by the recursive cone method for an even conjunction number *a*. It is known that the length of the square root of a natural number *n* can be constructed by a ruler and compass, as shown in Fig. 15 top. It can be done by starting from the diagonal of a square of side *1* and *n-1* nested recursive constructions.
A similar nested constructions from the "diagonal of a $1 \times 1/2$ rectangle" gives the same values of the convergence angle as obtained by the recursive cone method with odd conjuction number *a* in Table 1.
The convergence values of even and odd conjunction numbers together coincide with the list of "quadratic irrational numbers which are Pisot numbers" [5].

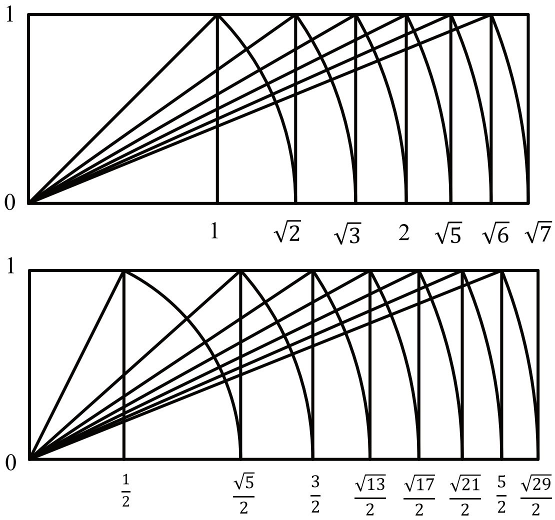


Figure 15.
Construction possible line lengths
with ruler and compass

9. Conclusion.

Can construct any regular polygon using the general recursive cone method and can construct any rational angle between *0* and *1*.
The general recursive cone method can be used to construct an angle which is the fractional part of square root of any natural number. The desired angle can be obtained by recursively repeating the {*a,b*} operation with two parameters: the even-conjunction number *a* and the *b*th central angle.
The general recursive cone method {*a,b*} operation is equivalent to an $[\overline{a}]_b$ continued fraction with a constant partial denominator *a* and a constant partial numerator *b*.
The angle determined by the {*a,b*} operation with odd-conjunction number *a* is expected to have similar geometric function as the golden angle.
The golden angle is the simplest angle that can be constructed by the general recursive cone method of odd conjunction numbers *a*.

Note

This paper is an intact machine translation of Hizume (2022, in Japanese)[6] . I did some proofreading, including unifying the terminology.

References

[1] As previous papers,
Akio hizume "Cone-pass," MANIFOLD #30 (2020).
Akio hizume "Cone-pass #2," ,MANIFOLD #31 (2020).
Akio hizume "Cone-pass," Bulletin of the Musashino Art University Vol. 51 (2021).
[2] Hiroaki Ito "Diophantine approximation by negative continued fraction," (2020).
[3] Ikuro's Homepage, In the article "The Problem of Continued Fraction Expansion (Part 2)," it is pointed out that in a regular continued fractions of the square root of any integer *n*, the integer part is *N* and the sequence of partial denominators is circular until *2N* appears, and that the sequence before *2N* is symmetrical.
This is the case in this paper where *a* is even. The same phenomenon is observed when *a* is an odd number.
[4] Akio hizume "Real Kecak System," MANIFOLD #2 (2001).
A ROM of the application software is included in the book "inter-native architecture OF music" (2006).
[5] Wikipedia "Pisot–Vijayaraghavan number."
[6] Akio Hizume "Generalization of Cone-pass and Continued Fraction" (in Japanese), MANIFOLD #33 (2022).